Handelman's theorem on polynomials with positive multiples

Valerio De Angelis and Selim Tuncel

Abstract

For a polynomial p in several variables and a face F of its Newton polytope, let p_F denote the polynomial consisting of the terms of p that lie in F, with the coefficients given by p. Handelman's theorem states that p has a polynomial multiple with positive coefficients if and only if no p_F has a zero with strictly positive coordinates. We give a short and self-contained account of its proof.

1 Introduction

Let $R = \mathbb{R}[x_1^{\pm}, \ldots, x_k^{\pm}]$ be the ring of Laurent polynomials in the variables $x_1, \ldots, x_k$, and let $R^+ = \mathbb{R}^+[x_1^{\pm}, \ldots, x_k^{\pm}]$ be the sub-semiring consisting of polynomials with nonnegative coefficients. For $a = (a_1, \ldots, a_k) \in \mathbb{Z}^k$, write $x^a = x_1^{a_1} \cdots x_k^{a_k}$ and denote the coefficient of x^a in $p \in R$ by p_a. Then $p = \sum_{a \in \mathbb{Z}^k} p_a x^a$ and p_a are nonzero for only finitely many $a \in \mathbb{Z}^k$. Let $\mathrm{Log}(p) = \{a \in \mathbb{Z}^k : p_a \neq 0\}$. The *Newton polytope* $N(p)$ is the rational convex hull of $\mathrm{Log}(p)$. Denote the collection of nonempty faces of $N(p)$ by $\mathcal{F}(p)$. For $F \in \mathcal{F}(p)$ let p_F be the sum of $p_a x^a$ over $a \in F \cap \mathrm{Log}(p)$, and call p_F the *F-face* of p. Note that in the case $F = N(p)$ we have $p_F = p$.

Handelman's Theorem [3]. *For $p \in R$ the following are equivalent.*

(i) There exists $q \in R$ such that $qp \in R^+ \setminus \{0\}$.

(ii) There exists $q \in R^+$ such that $\frac{p(1,\ldots,1)}{|p(1,\ldots,1)|} qp \in R^+ \setminus \{0\}$.

(iii) We have $p_F(\alpha) \neq 0$ for all $F \in \mathcal{F}(p)$ and $\alpha \in (0,\infty)^k$.

Call $p \in R$ *numerically positive* if it is the case that $p(\alpha) > 0$ for all $\alpha \in (0,\infty)^k$. Clearly, elements of R^+ are numerically positive. It is also easy to see that each of (i), (ii), (iii) implies that either p is numerically positive or $-p$ is numerically positive. The number $p(1,\ldots,1)/|p(1,\ldots,1)|$ in (ii) equals 1 or -1, depending on whether p or $-p$ is numerically positive.

Handelman's theorem has been applied in situations related to symbolic dynamics. (See [1, 4] for examples.) Its proof relies on a result from the theory of partially ordered Abelian

groups [2] and takes up a significant portion of a Memoir [3]. We will give a short and self-contained account. We emphasize that this is not a new proof but, rather, an economical account of the original proof. It will be clear that similar results are valid for polynomial rings, as well as Laurent polynomial rings, whether the coefficients are restricted to $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{Z}$.

2 Pure states

Let $f \in R^+$. Define S to be the subring of $R\left[\frac{1}{f}\right]$ generated by $\{\frac{x^a}{f} : a \in \mathrm{Log}(f)\}$, and let S^+ be the subsemiring generated by the same set. Then $S = S^+ - S^+$. For $g, h \in S$ we write $g \leq h$ if $h - g \in S^+$. Every element $g \in S$ is of the form $g = r/f^n$, with $n \in \mathbb{N}$, $r \in R$ and $\mathrm{Log}(r) \subset \mathrm{Log}(f^n)$. This expression is not unique since $r/f^n = rf^m/f^{n+m}$. An element r/f^n of S lies in S^+ if and only if $rf^m \in R^+$ for some $m \in \mathbb{N}$. The constant polynomial $1 = f/f$ belongs to S^+. It has the property that for every $\frac{r}{f^n} \in S$ there exists $K \in \mathbb{N}$ such that $\frac{r}{f^n} \leq K1$.

An additive group homomorphism $\phi : S \to \mathbb{R}$ will be called a *state* if $\phi(S^+) \subset \mathbb{R}^+$ and $\phi(1) = 1$. It is easy to see that states form a convex subset $\mathcal{S}$ of $\mathbb{R}^S$. An extreme point of $\mathcal{S}$ is said to be *pure*.

We have here examples of objects studied in the theory [2] of partially ordered Abelian groups: Let G be an (additive) Abelian group endowed with a translation invariant partial order $\leq$, and put $G^+ = \{g \in G : g \geq 0\}$. An element $u \in G^+$ is called an *order unit* if for every $g \in G$ there exists $K \in \mathbb{N}$ such that $g \leq Ku$. Given an order unit u, let $\mathcal{S}(G, u)$ denote the set of group homomorphisms $\phi : G \to \mathbb{R}$ with $\phi(G^+) \subset \mathbb{R}^+$ and $\phi(u) = 1$. An element $\phi \in \mathcal{S}(G, u)$ is a (normalized) *state*. Note, in particular, that states are order-preserving. The set $\mathcal{S}(G, u) \subset \mathbb{R}^G$ is convex, and its extreme points are said to be *pure*.

In the case we are considering, 1 is an order unit of S and $\mathcal{S} = \mathcal{S}(S, 1)$. Our first lemma is a special case of a general result [2]; we postpone its proof until section 4. We write $g > 0$ when $g \in S^+$ and $g \neq 0$.

Lemma 1 *Let $g \in S$. If $\phi(g) > 0$ for every pure state ϕ, then $g > 0$.*

Lemma 2 *Every pure state $\phi \in \mathcal{S}$ is multiplicative.*

Proof. Let ϕ be an extreme point of $\mathcal{S}$. Since $S = S^+ - S^+$ and ϕ is additive, it suffices to check that we have $\phi(gh) = \phi(g)\phi(h)$ for $g, h \in S^+$. Fix $g \in S^+$, and $K \in \mathbb{N}$ such that $g \leq K1$. Then $0 \leq \phi(g) \leq K$. For any $h \in S^+$ there exists $L \in \mathbb{N}$ with $h \leq L1$. If $\phi(g) = 0$ we have $0 \leq \phi(gh) \leq \phi(Lg) = L\phi(g) = 0$, and therefore $\phi(gh) = 0 = \phi(g)\phi(h)$. If $\phi(g) = K$ then, since h, $K1 - g \geq 0$, we have

$$0 \leq \phi((K1 - g)h) \leq \phi(L(K1 - g)) = L\phi(K1 - g) = 0,$$

so that

$$0 = \phi((K1 - g)h) = K\phi(h) - \phi(gh) = \phi(g)\phi(h) - \phi(gh).$$

Now assume $0 < \phi(g) < K$. Define $\psi \in \mathcal{S}$ by letting $\psi(h) = \phi(gh)/\phi(g)$ for $h \in S$. For $h \in S^+$ the inequality $\phi(gh) \le \phi(Kh)$ implies that, with $0 < t \equiv \phi(g)/K < 1$, we have $t\psi(h) \le \phi(h)$. This means that $\phi - t\psi \ge 0$ on S^+, so that $\theta = \frac{1}{1-t}(\phi - t\psi)$ is an element of $\mathcal{S}$. Since $\phi = t\psi + (1-t)\theta$ and ϕ is extreme, we conclude that $\psi = \phi$; that is, $\phi(gh) = \phi(g)\phi(h)$ for all $h \in S$. □

For pure $\phi \in \mathcal{S}$ define $L(\phi) = \{a \in \mathrm{Log}(f) : \phi(\frac{x^a}{f}) > 0\}$, and let $F(\phi)$ be the rational convex hull of $L(\phi)$.

Lemma 3 *For pure $\phi \in \mathcal{S}$ the set $F(\phi)$ is a face of the Newton polytope $N(f)$, and $L(\phi) = F(\phi) \cap \mathrm{Log}(f)$.*

Proof. Clearly $F(\phi) \subset N(f)$. Suppose $w \in F(\phi)$, $u, v \in N(f)$, $t \in (0,1) \cap \mathbb{Q}$ and $w = tu + (1-t)v$. We need to show $u, v \in F(\phi)$. Write

$$u = \sum_{a \in \mathrm{Log}(f)} \alpha_a a\,, \quad v = \sum_{a \in \mathrm{Log}(f)} \beta_a a\,, \quad w = \sum_{a \in L(\phi)} \gamma_a a\,,$$

with α_a, β_a, $\gamma_a \in \mathbb{Q}^+$ and $\sum_{a \in \mathrm{Log}(f)} \alpha_a = \sum_{a \in \mathrm{Log}(f)} \beta_a = \sum_{a \in L(\phi)} \gamma_a = 1$. Find $N \in \mathbb{N}$ such that all $\widetilde{\alpha}_a = Nt\alpha_a$, $\widetilde{\beta}_a = N(1-t)\beta_a$, $\widetilde{\gamma}_a = N\gamma_a$ are integers. Since ϕ is multiplicative (Lemma 2), we then have

$$\prod_{a \in L(\phi)} \left[\phi\left(\frac{x^a}{f}\right)\right]^{\widetilde{\gamma}_a} = \prod_{a \in \mathrm{Log}(f)} \left[\phi\left(\frac{x^a}{f}\right)\right]^{\widetilde{\alpha}_a} \cdot \prod_{a \in \mathrm{Log}(f)} \left[\phi\left(\frac{x^a}{f}\right)\right]^{\widetilde{\beta}_a}.$$

The left-hand side is positive by the definition of $L(\phi)$. So, the right-hand is also positive, and we have $\alpha_a = \beta_a = 0$ unless $a \in L(\phi)$. Hence $u, v \in F(\phi)$ and $F(\phi)$ is a face of $N(f)$.

It is clear that $L(\phi) \subset F(\phi) \cap \mathrm{Log}(f)$. For the reverse inclusion, suppose the element $w = \sum_{a \in L(\phi)} \gamma_a a$ above actually belongs to $\mathrm{Log}(f)$. Then we have

$$\left[\phi\left(\frac{x^w}{f}\right)\right]^N = \prod_{a \in L(\phi)} \left[\phi\left(\frac{x^a}{f}\right)\right]^{\widetilde{\gamma}_a}$$

and, since the right-hand side is positive, we conclude that $w \in L(\phi)$. □

Lemma 4 *Suppose $\phi \in \mathcal{S}$ is pure. Considering $L(\phi)$, the face $F(\phi)$ of $N(f)$ provided by lemma 3, and the $F(\phi)$-face $f_{F(\phi)}$ of f, there exists $\mu \in (0,\infty)^k$ such that for all $a \in L(\phi)$ we have*

$$\phi\left(\frac{x^a}{f}\right) = \frac{\mu^a}{f_{F(\phi)}(\mu)}\,.$$

We postpone the proof of lemma 4 to give that of the theorem. For $p \in R$ and $v \in \mathbb{R}^k$, let $\mathrm{in}_v(p)$ be the sum of $p_a x^a$ over those $a \in \mathrm{Log}(p)$ for which the dot product $a \cdot v$ is maximal. Observe that

$$\{\mathrm{in}_v(p) : v \in \mathbb{R}^k\} = \{p_F : F \in \mathcal{F}(p)\}.$$

Proof of Handelman's Theorem. It is clear that (ii) implies (i). Assume (i). Consider $F \in \mathcal{F}(p)$, and pick $v \in \mathbb{R}^k$ such that $\mathrm{in}_v(p) = p_F$. Then $\mathrm{in}_v(qp) = \mathrm{in}_v(q)\mathrm{in}_v(p) = \mathrm{in}_v(q)p_F$. Note that $\mathrm{in}_v(qp) \in R^+$ since $qp \in R^+$. It follows that for $\alpha \in (0,\infty)^k$ the product $\mathrm{in}_v(q)(\alpha)p_F(\alpha)$ is nonzero. Therefore, $p_F(\alpha) \neq 0$, and (i) implies (iii).

Now suppose (iii) holds. Taking $F = N(p)$ in (iii), we see that one of p, $-p$ is numerically positive. First suppose p is numerically positive. Let $F \in \mathcal{F}(p)$. Pick $v = (v_1, \ldots, v_k) \in \mathbb{R}^k$ such that $\mathrm{in}_v(p) = p_F$, and put $m = \max\{\mathrm{Log}(p) \cdot v\}$. Consider $\alpha = (\alpha_1, \ldots, \alpha_k) \in (0,\infty)^k$. Since $p(\alpha_1 e^{v_1 t}, \ldots, \alpha_k e^{v_k t}) = \sum_{a \in \mathrm{Log}(p)} p_a \alpha^a e^{(a \cdot v)t}$, we have

$$\mathrm{in}_v(p)(\alpha) = \lim_{t \to \infty} e^{-mt} p(\alpha_1 e^{v_1 t}, \ldots, \alpha_k e^{v_k t}),$$

and it follows from the numerical positivity of p that $p_F(\alpha) = \mathrm{in}_v(p)(\alpha) \geq 0$. Using (iii), we conclude that $p_F(\alpha) > 0$; that is, each p_F is numerically positive. Now pick any $f \in R^+$ such that $\mathrm{Log}(f) = \mathrm{Log}(p)$. We will show that $f^n p \in R^+$ for large $n \in \mathbb{N}$. Consider the ordered ring S associated above with f and, for a pure state $\phi \in \mathcal{S}$, let $F(\phi)$ be the face of $N(f) = N(p)$ discussed in lemmas 3 and 4. Observe that $p/f \in S$ and, letting $\mu \in (0,\infty)^k$ be as in the conclusion of lemma 4, use Lemmas 2, 3, 4 to calculate $\phi(\frac{p}{f})$:

$$\begin{aligned} \phi\left(\frac{p}{f}\right) &= \sum_{a \in \mathrm{Log}(p)} p_a \, \phi\left(\frac{x^a}{f}\right) = \sum_{a \in L(\phi)} p_a \, \phi\left(\frac{x^a}{f}\right) \\ &= \sum_{a \in L(\phi)} \frac{p_a \, \mu^a}{f_{F(\phi)}(\mu)} = \frac{p_{F(\phi)}(\mu)}{f_{F(\phi)}(\mu)} > 0\,. \end{aligned}$$

Combining this with lemma 1, we find that $\frac{p}{f} \in S^+$. Thus, in the case p is numerically positive, $f^n p \in R^+$ for large n. In the case $-p$ is numerically positive, apply this argument to $-p$ to see that $-f^n p \in R^+$ for large n. □

Note that the proof reveals that $\pm f^n p \in R^+$ for any $f \in R^+$ with $\mathrm{Log}(f) = \mathrm{Log}(p)$ and large enough n. It remains for us to prove lemmas 1 and 4.

3 Proof of lemma 4

For pure $\phi \in \mathcal{S}$, list the elements of $L(\phi)$ as $a_0, a_1, \ldots, a_m$. If $m = 0$ then $F(\phi) = L(\phi) = \{a_0\}$ and we take $\mu = (1, \ldots, 1)$ to obtain

$$\frac{\mu^{a_0}}{f_{F(\phi)}(\mu)} = \frac{1}{f_{a_0}} = \frac{\phi(1)}{f_{a_0}} = \frac{1}{f_{a_0}} \, \phi\left(\sum_{a \in \mathrm{Log}(f)} f_a x^a / f\right) = \phi\left(\frac{x^{a_0}}{f}\right),$$

as desired. Now suppose $m \geq 1$. Put $b_i = a_i - a_0$ for $i = 1, \ldots, m$. Consider the $k \times m$ matrix M with b_i as its i-th column, $i = 1, \ldots, m$. Let w be the vector in $\mathbb{R}^m$ whose i-th coordinate equals $\log \phi(\frac{x^{a_i}}{f}) - \log \phi(\frac{x^{a_0}}{f})$. Regarding M and w as linear maps $M : \mathbb{R}^m \to \mathbb{R}^k$ and $w : \mathbb{R}^m \to \mathbb{R}$, we show that $\mathrm{Ker}(M) \subset \mathrm{Ker}(w)$: Since M has integral entries, $\mathrm{Ker}(M)$ has a basis consisting of vectors in $\mathbb{Q}^m$ and it suffices to check that $\mathbb{Q}^m \cap \mathrm{Ker}(M) \subset \mathrm{Ker}(w)$. Suppose $t = (t_1, \ldots, t_m) \in \mathbb{Q}^m$ and $Mt = 0$. Then $\sum_{i=1}^m t_i b_i = 0$ and, putting $t_0 = \sum_{i=1}^m t_i$, we have $t_0 a_0 = \sum_{i=1}^m t_i a_i$. Find $N \in \mathbb{N}$ such that $Nt_i \in \mathbb{Z}$ for $i = 1, \ldots, m$. Letting $I^+ = \{1 \leq i \leq m : t_i \geq 0\}$, $I^- = \{1 \leq i \leq m : t_i < 0\}$, $t^+ = \sum_{i \in I^+} t_i$ and $t^- = \sum_{i \in I^-} |t_i|$, we find that

$$\left(\frac{x^{a_0}}{f}\right)^{Nt^-} \prod_{i \in I^+} \left(\frac{x^{a_i}}{f}\right)^{Nt_i} = \left(\frac{x^{a_0}}{f}\right)^{Nt^+} \prod_{i \in I^-} \left(\frac{x^{a_i}}{f}\right)^{N|t_i|}.$$

Applying ϕ and taking N-th roots,

$$\prod_{i=1}^m \left[\phi\left(\frac{x^{a_i}}{f}\right)\right]^{t_i} = \left[\phi\left(\frac{x^{a_0}}{f}\right)\right]^{t_0}.$$

That is, $\sum_{i=1}^m t_i \log \phi(\frac{x^{a_i}}{f}) = \sum_{i=1}^m t_i \log \phi(\frac{x^{a_0}}{f})$, which means that $wt = 0$. This shows $\mathrm{Ker}(M) \subset \mathrm{Ker}(w)$. It follows from this inclusion that the map $w : \mathbb{R}^m \to \mathbb{R}$ factors through M; that is, there exists $s = (s_1, \ldots, s_k) \in \mathbb{R}^k$ such that $w = sM$. Equivalently,

$$\log \phi\left(\frac{x^{a_i}}{f}\right) - \log \phi\left(\frac{x^{a_0}}{f}\right) = s \cdot a_i - s \cdot a_0.$$

Exponentiating and setting $\mu = (e^{s_1}, \ldots, e^{s_k})$, we find that

$$(*) \qquad \phi\left(\frac{x^{a_i}}{f}\right) / \phi\left(\frac{x^{a_0}}{f}\right) = \mu^{a_i - a_0}.$$

Hence

$$\begin{aligned} \frac{f_{F(\phi)}(\mu)}{\mu^{a_0}} &= \sum_{i=0}^m f_{a_i} \mu^{a_i - a_0} = \left[\phi\left(\frac{x^{a_0}}{f}\right)\right]^{-1} \sum_{i=0}^m f_{a_i} \phi\left(\frac{x^{a_i}}{f}\right) \\ &= \left[\phi\left(\frac{x^{a_0}}{f}\right)\right]^{-1} \sum_{a \in \mathrm{Log}(f)} f_a \phi\left(\frac{x^a}{f}\right) = \left[\phi\left(\frac{x^{a_0}}{f}\right)\right]^{-1} \phi(1), \end{aligned}$$

and, since $\phi(1) = 1$, we have $\phi(\frac{x^{a_0}}{f}) = \mu^{a_0} / f_{F(\phi)}(\mu)$. We then use $(*)$ again to see that $\phi(\frac{x^{a_i}}{f}) = \mu^{a_i} / f_{F(\phi)}(\mu)$ for $i = 0, 1, \ldots, m$. This completes the proof of lemma 4.

4 Proof of lemma 1

We now extract from [2] the proof of lemma 1. As we mentioned, lemma 1 is a special case of a general result; see pp. 61–65, 81–86, 95–96 of [2] for further details and general results.

Lemma 5 *Let G be a partially ordered Abelian group with order unit u and let H be a subgroup of G with $u \in H$. Let $g \in G$, $\theta \in \mathcal{S}(H,u)$, and set*

$$\begin{aligned} \alpha &= \sup\{\theta(h)/n : h \in H, n \in \mathbb{N}, h \leq ng\}, \\ \beta &= \inf\{\theta(h)/n : h \in H, n \in \mathbb{N}, ng \leq h\}. \end{aligned}$$

(a) We have $-\infty < \alpha \leq \beta < \infty$.

(b) If $\psi \in \mathcal{S}(H + \mathbb{Z}g, u)$ and ψ extends θ, then $\alpha \leq \psi(g) \leq \beta$.

(c) If $\alpha \leq \gamma \leq \beta$ then there exists $\psi \in \mathcal{S}(H + \mathbb{Z}g, u)$ such that ψ extends θ and $\psi(g) = \gamma$.

(d) The state $\theta \in \mathcal{S}(H,u)$ extends to an element of $\mathcal{S}(G,u)$.

Proof. Since u is an order unit, we can find K, $L \in \mathbb{N}$ with $-g \leq Ku$ and $g \leq Lu$. Then $-Ku \leq g \leq Lu$, which implies $-K \leq \alpha$, $\beta \leq L$. The rest of (a) and (b) are easily checked. To establish (c), first use the definition of α, β to observe that if $h \in H$ and $N \in \mathbb{Z}$ are such that $h + Ng \geq 0$ then $\theta(h) + N\gamma \geq 0$. It follows that if $h \in H$ and $N \in \mathbb{Z}$ satisfy $h + Ng = 0$ then both $\theta(h) + N\gamma \geq 0$ and $-\theta(h) - N\gamma \geq 0$ and, therefore, $\theta(h) + N\gamma = 0$. This means that θ extends to a group homomorphism $\psi : H + \mathbb{Z}g \to \mathbb{R}$ with $\psi(g) = \gamma$. The above observation shows that $\psi(h + Ng) \geq 0$ whenever $h + Ng \geq 0$. Finally, (d) follows from (c) by an application of Zorn's lemma. □

Proposition *Let G be a partially ordered group with order unit u, let $g \in G$, and set*

$$\begin{aligned} \alpha &= \sup\{m/n : m \in \mathbb{Z}, n \in \mathbb{N}, mu \leq ng\}, \\ \beta &= \inf\{m/n : m \in \mathbb{Z}, n \in \mathbb{N}, ng \leq mu\}. \end{aligned}$$

(a) We have $-\infty < \alpha \leq \beta < \infty$.

(b) If $\phi \in \mathcal{S}(G,u)$ then $\alpha \leq \phi(g) \leq \beta$.

(c) If $\alpha \leq \gamma \leq \beta$ then there exists $\phi \in \mathcal{S}(G,u)$ with $\phi(g) = \gamma$.

Proof. (a) and (b) are easy to check. For the proof of (c), put $H = \mathbb{Z}u$ and note that $\mathcal{S}(H,u)$ consists of a single element, θ. Observing that

$$\begin{aligned} \alpha &= \sup\{\theta(h)/n : h \in H = \mathbb{Z}u, n \in \mathbb{N}, h \leq ng\}, \\ \beta &= \inf\{\theta(h)/n : h \in H = \mathbb{Z}u, n \in \mathbb{N}, ng \leq h\}, \end{aligned}$$

apply lemma 5(c) to extend θ to $\psi \in \mathcal{S}(\mathbb{Z}u + \mathbb{Z}g, u)$ with $\psi(g) = \gamma$. Then apply lemma 5(d) to extend ψ to $\phi \in \mathcal{S}(G,u)$. □

Proof of lemma 1. First observe that if $\phi(g) > 0$ for all $\phi \in \mathcal{S}$ then $g > 0$: If $\phi(g) > 0$ for all $\phi \in \mathcal{S}$ then, by the above proposition, there exist $m, n \in \mathbb{N}$ such that $0 < m1 \leq ng$. It follows that $g > 0$.

Now equip $\mathbb{R}^S$ with the product topology, and $\mathcal{S} \subset \mathbb{R}^S$ with the induced topology. It is easy to see that $\mathbb{R}^S$ is then a locally convex topological vector space. For each $g \in S$, find $K_g \in \mathbb{N}$ such that $-K_g 1 \leq g \leq K_g 1$. Then $\mathcal{S} \subset \prod_{g \in S}[-K_g, K_g]$ and, using Tychonoff's theorem, we see that $\mathcal{S}$ is a compact convex subset of $\mathbb{R}^S$. In addition, for any $g \in S$, the map $\phi \mapsto \phi(g) : \mathcal{S} \to \mathbb{R}$ is affine and continuous. It is a well-known corollary of the Krein-Milman theorem (see pp. 81–86 of [2]) that an affine continuous map on a compact convex subset $\mathcal{S}$ of a locally convex topological vector space achieves its infimum at an extreme point of $\mathcal{S}$. So, if $\phi(g) > 0$ for all pure $\phi \in \mathcal{S}$ then the initial observation of the proof applies and shows $g > 0$. □

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