The Minimal Polynomials of $\sin(2\pi/p)$ and $\cos(2\pi/p)$

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Every student of trigonometry knows that if $n = 1, 2, 3, 4$ or 6, then $\cos(2\pi/n)$ is a rational number, and so it is the root of a first degree polynomial with integer coefficients. Another way of expressing this is to say that for the values of n listed above, $\cos(2\pi/n)$ is an *algebraic number* with *algebraic degree* one. Similarly well-known is the fact that for $n = 5, 8$, or 12, $\cos(2\pi/n)$ is a root of a quadratic, irreducible polynomial with integer coefficients, and we express this by saying that for $n = 5, 8$, or 12, $\cos(2\pi/n)$ is an algebraic number of algebraic degree two. (The case $n = 5$ may not be as popular as the others; for those who wonder, $\cos(2\pi/5)$ is a root of the polynomial $4x^2 + 2x - 1$).

As the reader will no doubt have guessed at this point, if a real (or complex) number is a root of an irreducible polynomial of degree n and with integer coefficients, we say that it is an algebraic number with algebraic degree n. The irreducible polynomial in question is its *minimal polynomial.* So the previous paragraph can be rephrased by saying that the algebraic degree of $\cos(2\pi/n)$ for $n = 1, 2, 3, 4, 5, 6, 8$ and 12 is widely known.

But probably not all trigonometry students know that $1, 2, 3, 4$ and 6 are the *only* positive integer values of n for which the algebraic degree of $\cos(2\pi/n)$ is one, and $5, 8$ and 12 are the only values for which it is two. In fact, there is a general formula to compute the algebraic degree of $\cos(2\pi/n)$ and $\sin(2\pi/n)$ (see the remark at the end).

The purpose of this note is to exhibit explicit expressions for the minimal polynomials of $\cos(2\pi/n)$ and $\sin(2\pi/n)$ in the special case that n is a prime.

We will show that if $p > 2$ is a prime number, then the minimal polynomial of $\sin(2\pi/p)$ is

$$S_p(x) = \sum_{k=0}^{(p-1)/2} (-1)^k \begin{pmatrix} p \\ 2k+1 \end{pmatrix} \left(1 - x^2\right)^{(p-1)/2-k} x^{2k},$$

of degree $p - 1$, and the minimal polynomial of $\cos(2\pi/n)$ is

$$C_p(x) = S_p\left(\sqrt{\frac{1-x}{2}}\right),$$

of degree $(p-1)/2$. All arguments are elementary, and make no use of field theory. The main tool is Eisenstein's criterion for irreducibility of polynomials, which we recall below.

Eisenstein's criterion [1, p.160] *Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with integer coefficients. If there is a prime p that divides each of $a_0, a_1, \ldots, a_{n-1}$, while p does not divide a_n, and p^2 does not divide a_0, then $f(x)$ is irreducible.*

We now proceed to the derivation of $S_p(x)$. Let $p > 2$ be a prime number. Using Euler's identities $e^{i\theta} = \cos\theta + i\sin\theta$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, and setting

$x = \sin\theta$, we derive an expression for $\dfrac{\sin(p\theta)}{\sin\theta}$ in terms of x. We have

$$\frac{\sin(p\theta)}{\sin\theta} = \frac{1}{2ix}\left[\left(\sqrt{1-x^2}+ix\right)^p - \left(\sqrt{1-x^2}-ix\right)^p\right],$$

and using the binomial theorem to write

$$\left(\sqrt{1-x^2} \pm ix\right)^p = \sum_{k=0}^{p} (\pm i)^k \binom{p}{k} \left(1-x^2\right)^{(p-k)/2} x^k$$

in the expression in brackets, we find that the even terms of the sum will cancel, and we obtain

$$\frac{\sin(p\theta)}{\sin\theta} = \sum_{k=0}^{(p-1)/2} (-1)^k \binom{p}{2k+1} \left(1-x^2\right)^{(p-1)/2-k} x^{2k} = S_p(x). \qquad (*)$$

Note that $S_p(x)$ is a polynomial in x^2 with integer coefficients, and of degree $p-1$. This polynomial is closely related to the classical Chebyshev polynomials of the second kind $U_n(x)$, defined by $U_n(\cos\theta) = \dfrac{\sin((n+1)\theta)}{\sin\theta}$, because $S_p(x) = U_{p-1}\left(\sqrt{1-x^2}\right)$. From $(*)$, we have $S_p(\sin\theta) = \dfrac{\sin(p\theta)}{\sin\theta}$ for all θ, and so $S_p(\sin(2\pi/p)) = \dfrac{\sin(2\pi)}{\sin(2\pi/p)} = 0$. Hence to show that $S_p(x)$ is the minimal polynomial of $\sin(2\pi/p)$ we only need to prove that $S_p(x)$ is irreducible. Clearly $S_p(0) = p$, so that the condition on the constant term required by Eisenstein's criterion is satisfied. If $k < \dfrac{p-1}{2}$, then the binomial coefficient $\dbinom{p}{2k+1} = \dfrac{p(p-1)\cdots(p-2k)}{(2k+1)!}$ is a multiple of p, because all prime factors of $(2k+1)!$ are less than p, so that $\dfrac{(p-1)\cdots(p-2k)}{(2k+1)!}$ must be an integer. Since the term corresponding to $k = \dfrac{p-1}{2}$ in $(*)$ is $(-1)^{(p-1)/2} x^{p-1}$, it does not contribute to any term of degree less than $p-1$.

So we conclude that all coefficients of $S_p(x)$ other than the leading one are divisible by p. Since the degree of $S_p(x)$ is $p-1$, $x^{p-1}S_p\left(\frac{1}{x}\right)$ is a polynomial in x whose constant term is the leading coefficient of $S_p(x)$. We find from $(*)$ that $x^{p-1}S_p\left(\frac{1}{x}\right) = \sum_{k=0}^{(p-1)/2}(-1)^k\begin{pmatrix} p \\ 2k+1 \end{pmatrix}(x^2-1)^{(p-1)/2-k}$, and so the leading coefficient of $S_p(x)$ is $(-1)^{(p-1)/2}\sum_{k=0}^{(p-1)/2}\begin{pmatrix} p \\ 2k+1 \end{pmatrix}$. To evaluate this sum, note that $2^p = \sum_{k=0}^{p}\begin{pmatrix} p \\ k \end{pmatrix} = \sum_{k=0}^{(p-1)/2}\begin{pmatrix} p \\ 2k+1 \end{pmatrix} + \sum_{k=0}^{(p-1)/2}\begin{pmatrix} p \\ 2k \end{pmatrix} = \sum_{k=0}^{(p-1)/2}\begin{pmatrix} p \\ 2k+1 \end{pmatrix} + \sum_{k=0}^{(p-1)/2}\begin{pmatrix} p \\ p-2k-1 \end{pmatrix} = 2\sum_{k=0}^{(p-1)/2}\begin{pmatrix} p \\ 2k+1 \end{pmatrix}$. So the leading coefficient of $S_p(x)$ is $(-1)^{(p-1)/2}\,2^{p-1}$, and from Eisenstein's criterion we conclude that $S_p(x)$ is irreducible, and therefore it is the minimal polynomial of $\sin(2\pi/p)$.

Now consider $C_p(x) = S_p\left(\sqrt{\frac{1-x}{2}}\right)$, of degree $\frac{p-1}{2}$ (recall that $S_p(x)$ is a polynomial in x^2, so $S_p(\sqrt{x})$ is a polynomial in x). We then have $C_p(\cos\theta) = S_p(\sin(\theta/2)) = \frac{\sin(p\theta/2)}{\sin(\theta/2)}$, and so $C_p(\cos(2\pi/p)) = \frac{\sin(\pi)}{\sin(\pi/p)} = 0$. It is a simple exercise in the use of trigonometric identities to show (by induction, for example) that for each positive integer m, we have $\frac{\sin[(m+1/2)\theta]}{\sin(\theta/2)} = 1 + 2\sum_{k=1}^{m}\cos(k\theta)$. Using this with $m = \frac{p-1}{2}$, we find that $C_p(\cos\theta) = 1 + 2\sum_{k=1}^{(p-1)/2}\cos(k\theta)$. The terms of this sum define the Chebyshev polynomials of the first kind $T_k(x)$, given by $T_k(\cos\theta) = \cos(k\theta)$, which are easily seen to have integer coefficients. So we conclude that $C_p(x)$ has integer coefficients, and in order to prove that $C_p(x)$ is the minimal polynomial of $\cos(2\pi/p)$ it only remains to show that it is irreducible. If $f(x)$ were a factor of $C_p(x)$, then $f(1-2x)$ would be a factor of $S_p(\sqrt{x})$. But the coefficient of x^k in $S_p(\sqrt{x})$ is the coefficient of x^{2k} in $S_p(x)$, so the same application of Eisenstein's criterion as before shows that $S_p(\sqrt{x})$ is irreducible. This concludes the derivation of the minimal polynomials mentioned in the title.

We remark that the polynomials $S_p(x)$ and $C_p(x)$ are also primitive, in the sense that the greatest common factor of their coefficients is one. This is not guaranteed by Eisenstein's criterion (as can be seen, for example, if we multiply the polynomials by 2), but it is easily checked for $S_p(x)$ because the constant term is p while the leading coefficient is a power of 2, and for $C_p(x)$ because $C_p(0) = C_p(\cos(\pi/2)) = \dfrac{\sin(p\pi/4)}{\sin(\pi/4)} = \pm 1$.

We list below $S_p(x)$ and $C_p(x)$ for the first few values of p:

$$
\begin{aligned}
S_3(x) &= -4x^2 + 3 \\
S_5(x) &= 16x^4 - 20x^2 + 5 \\
S_7(x) &= -64x^6 + 112x^4 - 56x^2 + 7 \\
S_{11}(x) &= -1024x^{10} + 2816x^8 - 2816x^6 + 1232x^4 - 220x^2 + 11 \\
S_{13}(x) &= 4096x^{12} - 13312x^{10} + 16640x^8 - 9984x^6 + 2912x^4 - 364x^2 + 13 \\
C_3(x) &= 2x + 1 \\
C_5(x) &= 4x^2 + 2x - 1 \\
C_7(x) &= 8x^3 + 4x^2 - 4x - 1 \\
C_{11}(x) &= 32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1 \\
C_{13}(x) &= 64x^6 + 32x^5 - 80x^4 - 32x^3 + 24x^2 + 6x - 1
\end{aligned}
$$

Remark: The general formula to compute the algebraic degree of $\sin(2\pi/n)$ and $\cos(2\pi/n)$ in terms of Euler's totient function ϕ for all $n > 2$ is as follows [2, p.289]:

1. $\deg(\cos(2\pi/n)) = \frac{1}{2}\phi(n)$

2. If $n \neq 4$, and we write $n = 2^r m$, where m is odd, then
$$\deg(\sin(2\pi/n)) = \begin{cases} \phi(n) & \text{if } r = 0 \text{ or } 1 \\ \frac{1}{4}\phi(n) & \text{if } r = 2 \\ \frac{1}{2}\phi(n) & \text{if } r \geq 3 \end{cases}.$$

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References

1. I.N. Herstein, *Topics in Algebra*, Xerox College Publishing, Lexington, MA, 1975, Second edition.

2. Paulo Ribenboim, *Algebraic Numbers*, Wiley-Interscience, 1972.