COMPLEMENTARY BELL NUMBERS: ARITHMETICAL PROPERTIES AND WILF'S CONJECTURE

TEWODROS AMDEBERHAN, VALERIO DE ANGELIS, AND VICTOR H. MOLL

A tribute to Herb Wilf's birthday

Abstract. The 2-adic valuations of Bell and complementary Bell numbers is determined. The complementary Bell numbers are known to be zero at $n = 2$ and H. S. Wilf conjectured that this is the only case where vanishing occurs. N. C. Alexander and J. An prove (independently) that there are at most two indices where this happens. This paper presents yet an alternative proof of the latter.

1. Introduction

The Stirling numbers of second kind $S(n,k)$, defined for $n \in \mathbb{N}$ and $0 \leq k \leq n$, count the number of ways to partition a set of n elements into exactly k nonempty subsets (blocks). The *Bell numbers*

$$B(n) = \sum_{k=0}^{n} S(n,k) \tag{1.1}$$

count all such partitions independent of size and the *complementary Bell numbers*

$$\widetilde{B}(n) = \sum_{k=0}^{n} (-1)^k S(n,k) \tag{1.2}$$

takes the parity of the number of blocks into account. The exponential generating functions are given by

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \exp(\exp(x) - 1) \text{ and } \sum_{n=0}^{\infty} \widetilde{B}(n) \frac{x^n}{n!} = \exp(1 - \exp(x)). \tag{1.3}$$

In this paper we consider arithmetical properties of the Bell and complementary Bell numbers. The results described here are part of a general program to describe properties of p-adic valuations of classical sequences. The example of Stirling numbers is described in [3], the ASM numbers that count the number of Alternating Sign Matrices appear in [14] and a not-so-classical sequence appearing in the evaluation of a rational integral is described in [2, 9]. On the other hand, much of our interest in the valuations of the complementary Bell numbers is motivated by

Wilf's conjecture: $\widetilde{B}(n) = 0$ only for $n = 2$.

Date: August 2, 2011.

1991 *Mathematics Subject Classification.* Primary 11B50, Secondary 05A15.

Key words and phrases. valuations, Bell numbers, complementary Bell numbers, closed-form summation.

The guiding strategy for us is this: if we manage to prove that $\nu_2(\widetilde{B}(n))$ is finite for $n > 2$, the non-vanishing result will follow. The authors [4] have succeeded in employing this method to prove that the sequence

$$x_n = \frac{n + x_{n-1}}{1 - n x_{n-1}}, \text{ starting at } x_1 = 1 \tag{1.4}$$

only vanishes at $n = 3$. The more natural question that $x_n \notin \mathbb{Z}$ for $n > 5$ remains open.

The following notation is adopted throughout this paper: for $n \in \mathbb{N}$ and a prime p, the *p-adic valuation of* n, denoted by $\nu_p(n)$, is the largest power of p that divides n. The value $\nu_p(0) = +\infty$ is consistent with the fact that any power of p divides 0. As an example, the complementary Bell number $\widetilde{B}(14) = 110176$ factors as $2^5 \cdot 11 \cdot 313$, therefore $\nu_2(\widetilde{B}(14)) = 5$ and $\nu_3(\widetilde{B}(14)) = 0$. Legendre established the formula

$$\nu_p(n) = \frac{n - s_p(n)}{p - 1} \tag{1.5}$$

where $s_p(n)$ is the sum of the digits of n in base p.

The exponential generating function (1.3) and the series representation

$$\widetilde{B}(n) = e \sum_{r=0}^{\infty} (-1)^r \frac{r^n}{r!}, \tag{1.6}$$

as well as elementary properties of the complementary Bell numbers are presented in [15]. The numbers $\widetilde{B}(n)$ also appear in the literature as the Uppuluri-Carpenter numbers. Subbarao and Verma [13] established asymptotic growth of $\widetilde{B}(n)$, showing that

$$\limsup_{n\to\infty} \frac{\log|\widetilde{B}(n)|}{n \log n} = 1. \tag{1.7}$$

The non-vanishing of $\widetilde{B}(n)$ has been considered by M. Klazar [7, 8] in the context of partitions, by M. R. Murty [10] in reference to p-adic irrationality. Y. Yang [16] established the result $|\{n \le x : \widetilde{B}(n) = 0\}| = O(x^{2/3})$ and De Wannemacker et al [12] proved that if $n \equiv 2,\ 2944838 \bmod 3 \cdot 2^{20}$, then $\widetilde{B}(n) \neq 0$. The main result of this paper is that $\widetilde{B}(n) = 0$ has at most two solutions. This has been achieved by different techniques by N. C. Alexander [1] and Junkyu An [5]. Our interest in the non-vanishing questions comes from the theory of summation in finite terms. The methods developed by R. Gosper show that the finite sum

$$\sum_{k=1}^{n} k! \tag{1.8}$$

does not admit a closed-form expression as a hypergeometric function of n. The identity

$$\sum_{k=1}^{n} k^a k! = (-1)^{a+1} \sum_{\ell=0}^{a} (n+\ell)! r_\ell + (-1)^{a+1} \widetilde{B}(a+1) \sum_{k=1}^{n} k! \tag{1.9}$$

where

$$r_\ell = \sum_{i=\ell+1}^{a} (-1)^i S(a+1,i), \tag{1.10}$$

shows that Wilf's conjecture, in the positive, implies that the identity

$$\sum_{k=1}^{n} kk! = (n+1)! - 1 \tag{1.11}$$

is unique in this category. M. Petkovsek, H. S. Wilf and D. Zeilberger [11] is the standard reference for issues involving closed-form summation. The details for (1.9) are provided in [6].

Section 2 presents a family of polynomials that play a crucial role in the study of the 2-adic valuations of Bell numbers given in Section 3. The main arguments in the work presented here are based on the representation of the polynomials introduced in Section 2 in terms of ascending and descending factorials. This is discussed in Section 4. An alternative proof of the analytic expressions for the valuations of regular Bell numbers is presented in Section 5. This serves as a motivating example for the more difficult case of the 2-adic valuations of complementary Bell numbers. Experimental data on these valuations are presented in Section 6. The data suggests that only those indices congruent to 2 modulo 3 need to be considered. The study of this case begins in Section 7 where these valuations are determined for all but two classes modulo 24. The two remaining classes require the introduction of an infinite matrix. This is done in Section 8. The two remaining classes are analyzed in Sections 9 and 10, respectively. The final section presents the exponential generating functions of the two classes of polynomials employed in this work, and some open problems.

2. An auxiliary family of polynomials

The recurrence for the Stirling numbers of second kind

$$S(n+1,k) = S(n,k-1) + kS(n,k) \tag{2.1}$$

is summed over $0 \leq k \leq n+1$ to produce

$$\sum_{k=0}^{n+1} S(n+1,k) = \sum_{k=0}^{n} (k+1)S(n,k) \tag{2.2}$$

using the vanishing of $S(n,k)$ for $k < 0$ or $k > n$. Iteration of this procedure leads to the next result.

Lemma 2.1. *The family of polynomials $\mu_j(k)$, defined by*

$$\mu_{j+1}(k) = k\mu_j(k) + \mu_j(k+1), \tag{2.3}$$

$$\mu_0(k) = 1, \tag{2.4}$$

satisfy

$$B(n+j) = \sum_{k=0}^{n+j} S(n+j,k) = \sum_{k=0}^{n} \mu_j(k)S(n,k), \tag{2.5}$$

for all n, $j \geq 0$.

Proof. The proof is by induction on j. The inductive step gives

$$\sum_{k=0}^{(n+1)+j} S((n+1)+j,k) = \sum_{k=0}^{n+1} \mu_j(k) S(n+1,k). \tag{2.6}$$

The recurrence (2.1) and (2.3) yield the result. □

Note. The polynomials $\mu_j(k)$ have positive integer coefficients and the first few are given by

$$\begin{aligned} \mu_0(k) &= 1 \\ \mu_1(k) &= k+1 \\ \mu_2(k) &= k^2+2k+2 \\ \mu_3(k) &= k^3+3k^2+6k+5. \end{aligned}$$

The degree of μ_j is j so the family $Z_m := \{\mu_j : 0 \le j \le m\}$ forms a basis for the space of polynomials of degree at most m.

The special polynomial

$$\begin{aligned} \mu_{12}(k) &= k^{12}+12k^{11}+132k^{10}+1100k^9+7425k^8+41184k^7 \\ &\quad +187572k^6+694584k^5+2049300k^4+4652340k^3 \\ &\quad +7654350k^2+8142840k+4213597 \end{aligned} \tag{2.7}$$

plays a crucial role in the study of 2-adic valuation of Bell numbers discussed in Section 3.

3. The 2-adic valuation of Bell numbers

In this section we determine the 2-adic valuation of the Bell numbers. The data presented in Figure 1 suggests to examining this valuation along classes modulo 12.

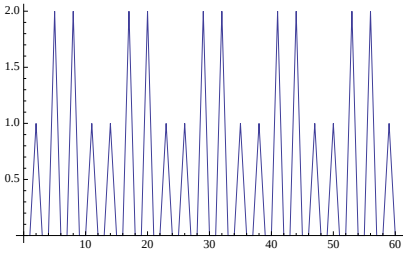

FIGURE 1. The 2-adic valuation of Bell numbers

Theorem 3.1. *The* 2*-adic valuation of the Bell numbers satisfy*

$$\nu_2(B(n)) = 0 \quad \text{if } n \equiv 0,\, 1 \bmod 3. \tag{3.1}$$

In the missing case, $n \equiv 2 \bmod 3$*, the sequence* $\nu_2(B(3n+2))$ *is a periodic function of period* 4*. The repeating values are* $\{1,\, 2,\, 2,\, 1\}$*. In particular, the* 2*-adic valuation*

of the Bell numbers is completely determined modulo 12. *In detail,*

$$\nu_2(B(12n+j)) \quad = \quad \begin{cases} 0 & \text{if } j \equiv 0,\, 1\ 3,\, 4,\, 6,\, 7,\, 9,\, 10 \mod 12 \\ 1 & \text{if } j \equiv 2,\, 11 \mod 12 \\ 2 & \text{if } j \equiv 5,\, 8 \mod 12. \end{cases} \tag{3.2}$$

The proof of the theorem starts with a congruence for the Bell numbers.

Lemma 3.2. *The Bell numbers satisfy*

$$B(n+24) \equiv B(n) \bmod 8. \tag{3.3}$$

Proof. The identity (2.5) gives

$$\sum_{k=0}^{n+12} S(n+12,k) = \sum_{k=0}^{n} \mu_{12}(k) S(n,k). \tag{3.4}$$

The polynomial $\mu_{12}(k)$ given in (2.7) is now expressed in terms of the basis of ascending factorials

$$(k)^{[m]} := k(k+1)(k+2)\cdots(k+m-1),\ m \in \mathbb{N},\ \text{with } (k)^{[0]} = 1. \tag{3.5}$$

A direct calculation shows that

$$\mu_{12}(k) \equiv \sum_{m=0}^{12} a_m (k)^{[m]} \tag{3.6}$$

with $a_0 = 421359 \equiv 5$, $a_1 = 3633280 \equiv 0$, $a_2 = 1563508 \equiv 4$, and $a_3 = 414920 \equiv 0 \bmod 8$. Also, for $m \geq 4$, we have $(k)^m \equiv 0 \bmod 8$. Thus

$$\mu_{12}(k) \equiv 5 + 4k(k+1) \equiv 5 \bmod 8. \tag{3.7}$$

Now (3.4) produces

$$\sum_{k=0}^{n+12} S(n+12,k) \equiv 5 \sum_{k=0}^{n} S(n,k) \bmod 8, \tag{3.8}$$

that is, $B(n+12) \equiv 5B(n) \bmod 8$. Repeating this yields $B(n+24) \equiv 5B(n+12) \equiv 25B(n) \equiv B(n) \mod 8$. □

The result of the theorem now follows computating of the first 24 Bell numbers modulo 8 to obtain the pattern asserted in the theorem.

Remark 3.3. The p-adic valuation of Bell numbers for primes $p \neq 2$ exhibit some patterns. Figure 2 shows the case $p = 3$.

Experimental observations show that, if $j \not\equiv 2 \bmod 3$, then

$$\nu_3(B_{12(n+j)+j}) = \nu_3(B_{12n}),\ \text{for } n \geq 0. \tag{3.9}$$

In other words, up to a shift, the valuations $\nu_3(B_{12n+j})$ are independent of j.

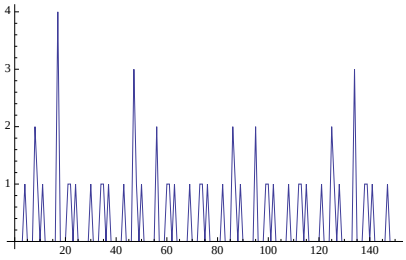

FIGURE 2. The 3-adic valuation of Bell numbers

4. A representation in two basis

The set

$$Z_m = \{\mu_j(k) : 0 \leq j \leq m\} \tag{4.1}$$

is a basis of the vector space of polynomials of degree at most m. This section explores the representation of this basis in terms of the usual *ascending factorials*, defined by

$$\begin{aligned} (k)^{[r]} &:= k(k+1)(k+2)\cdots(k+r-1) \quad \text{for } r>0, \\ (k)^{[0]} &:= 1, \end{aligned} \tag{4.2}$$

and the *descending factorials*, given by

$$\begin{aligned} (k)_r &:= k(k-1)(k-2)\cdots(k-r+1) \quad \text{for } r>0, \\ (k)_0 &:= 1, \end{aligned} \tag{4.3}$$

Definition 4.1. The coefficients of $\mu_n(r)$ with respect to these bases are denoted

$$\mu_j(k) = \sum_{r=0}^{j} a_j(r)(k)^{[r]} \quad \text{and } \mu_j(k) = \sum_{r=0}^{j} d_j(r)(k)_r. \tag{4.4}$$

These coefficients are stored in the vectors

$$\mathbf{a_j} := [a_j(0),\, a_j(1),\, \cdots] \quad \text{and} \quad \mathbf{d_j} := [d_j(0),\, d_j(1),\, \cdots] \tag{4.5}$$

where $a_j(r) = d_j(r) = 0$ for $r > j$.

Certain properties of $(k)_r$ and $(k)^{[r]}$ required in the analysis of the 2-adic valuations are stated below.

Lemma 4.2. *The ascending factorial symbol satisfies*

$$\begin{aligned} (k-1)^{[r]} &= (k)^{[r]} - r(k)^{[r-1]} \\ k(k)^{[r]} &= (k)^{[r+1]} - r(k)^{[r]}. \end{aligned}$$

The corresponding relations for the descending factorials are

$$\begin{aligned} (k+1)_r &= (k)_r + r(k)_{r-1} \\ k(k)_r &= (k)_{r+1} + r(k)_r. \end{aligned}$$

The next step is to transform the recurrence for μ_j in (2.3) into recurrences for the coefficients $a_j(r)$ and $d_j(r)$.

Proposition 4.3. *The coefficients $a_j(r)$ in Definition 4.1 satisfy*

$$a_{j+1}(r) - (r+1)a_{j+1}(r+1) = a_j(r-1) - 2ra_j(r) + (r+1)^2 a_j(r+1), \tag{4.6}$$

with the assumptions that $a_j(r) = 0$ if $r < 0$ or $r > j$.

Proof. This follows directly from the recurrence for μ_j and the properties described in Lemma 4.2. □

Note. The recurrences for the coefficients $\mathbf{a_j}$ can be written using the (infinite) matrices

$$\mathbf{M} = (m_{ij})_{i,\,j\geq 0} \quad \text{and } \mathbf{N} = (n_{ij})_{i,\,j\geq 0} \tag{4.7}$$

with

$$m_{ij} = \begin{cases} 1 & \text{if } i = j \\ -(i+1) & \text{if } i = j-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad n_{ij} = \begin{cases} 1 & \text{if } i = j+1 \\ -2(i-1) & \text{if } i = j \\ i^2 & \text{if } i = j-1 \\ 0 & \text{otherwise} \end{cases}$$

in the form

$$\mathbf{M a_{j+1}} = \mathbf{N a_j}. \tag{4.8}$$

The analogue of Proposition (4.3) for descending factorials is stated next.

Proposition 4.4. *The coefficients $d_j(r)$ in (4.1) satisfy*

$$d_{j+1}(r) = d_j(r-1) + (r+1)d_j(r) + (r+1)d_j(r+1), \tag{4.9}$$

with the assumptions that $d_j(r) = 0$ if $r < 0$ or $r > j$.

Note. The recurrence for $\mathbf{d_j}$ is now written using $\mathbf{T} = (t_{ij})_{i,\,j\geq 0}$, where

$$t_{ij} = \begin{cases} i+1 & \text{if } i = j \\ i & \text{if } i = j-1 \\ 1 & \text{if } i = j+1 \\ 0 & \text{otherwise} \end{cases}$$

in the form

$$\mathbf{d_{j+1}} = \mathbf{T d_j}. \tag{4.10}$$

5. An alternative approach to valuation of Bell numbers

This section presents an alternative proof of the congruence (3.2) based on the results of Section 4. Recall that this congruence provides a complete structure of the 2-adic valuation of the Bell numbers. The ideas introduced here provide a partial description of the 2-adic valuations of complementary Bell numbers.

The first step is to identify the Bell numbers as the first entry of the vectors $\mathbf{a}_j$ and $\mathbf{d}_j$.

Lemma 5.1. *The Bell numbers are given by*

$$B(j) = \mu_j(0) = a_j(0) = d_j(0). \tag{5.1}$$

Proof. Let $n = 0$ in the identity (2.5) to obtain $B(j) = \mu_j(0)$. The other two expressions for the Bell numbers $B(j)$ are obtained by letting $k = 0$ in (4.4). □

The congruence for the Bell numbers now arises from the analysis of the relations (4.8) and (4.10) modulo 8. The key statement is provided next.

Lemma 5.2. *If $k \in \mathbb{N}$ and $r \geq 4$, then*

$$(k)^{[r]} \equiv (k)_r \equiv 0 \bmod 8. \tag{5.2}$$

Proof. Among any set of four consecutive integers there is one that is a multiple of 2 and a different one that is a multiple of 4. □

The system (4.8) now reduces to

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{j+1}(0) \\ a_{j+1}(1) \\ a_{j+1}(2) \\ a_{j+1}(3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 4 & 0 \\ 0 & 1 & -4 & 9 \\ 0 & 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} a_j(0) \\ a_j(1) \\ a_j(2) \\ a_j(3) \end{bmatrix}.$$

Inverting the matrix on the left and taking entries modulo 8 leads to

$$\mathbf{a}_{\mathbf{j+1}}^{(\mathbf{4})} \equiv X_4 \mathbf{a}_{\mathbf{j}}^{(\mathbf{4})} \bmod 8 \tag{5.3}$$

where $\mathbf{a}_{\mathbf{j}}^{(\mathbf{4})}$ represents the first four entries of the coefficient vector $\mathbf{a_j}$ and

$$X_4 = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 1 & 0 & 2 & 6 \\ 0 & 1 & 7 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Now observe that

$$\mathbf{a}_{\mathbf{j+2}}^{(\mathbf{4})} \equiv X_4 \mathbf{a}_{\mathbf{j+1}}^{(\mathbf{4})} \equiv X_4^2 \mathbf{a}_{\mathbf{j}}^{(\mathbf{4})} \bmod 8 \tag{5.4}$$

and this extends to

$$\mathbf{a}_{\mathbf{j+s}}^{(\mathbf{4})} \equiv X_4^s \mathbf{a}_{\mathbf{j}}^{(\mathbf{4})} \bmod 8 \tag{5.5}$$

for any $s \in \mathbb{N}$.

Lemma 5.3. *The matrix X satisfies $X^{24} \equiv I \bmod 8$.*

Proof. Direct (symbolic) calculation. □

The Bell number $B(j)$ is the first entry of the vector $\mathbf{a}_{\mathbf{j}}^{(\mathbf{4})}$. Then considering the first entry in the relation

$$\mathbf{a}_{\mathbf{j+24}}^{(\mathbf{4})} \equiv X_4^{24} \mathbf{a}_{\mathbf{j}}^{(\mathbf{4})} \bmod 8 \tag{5.6}$$

gives the congruence $B(j+24) \equiv B(j) \bmod 8$.

Note. The corresponding relation for the coefficient vector $\mathbf{d_j}$ is simpler: the system (4.10) reduces to

$$\begin{bmatrix} d_{j+1}(0) \\ d_{j+1}(1) \\ d_{j+1}(2) \\ d_{j+1}(3) \end{bmatrix} \equiv T_4 \times \begin{bmatrix} d_j(0) \\ d_j(1) \\ d_j(2) \\ d_j(3) \end{bmatrix} \bmod 8 \tag{5.7}$$

where

$$T_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}. \tag{5.8}$$

The matrix T_4 also satisfies $T_4^{24} \equiv I \bmod 8$ and the argument proceeds as before.

6. Some experimental data on $\nu_2(\widetilde{B}(n))$

This section discusses the 2-adic valuations of the complementary Bell numbers $\widetilde{B}(n)$. The data is depicted in Figure 3 in the range $3 \leq n \leq 1000$.

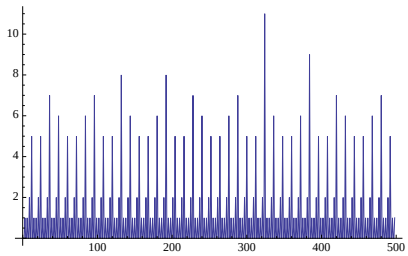

Figure 3. The 2-adic valuation of the complementary Bell numbers

This discussion begins with some empirical data from the sequence $\nu_2(\widetilde{B}(n))$. For $3 \leq n \leq 30$, the list is

$$\{0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 5, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 5, 0, 0, 1, 0\}. \tag{6.1}$$

This suggests that $\nu_2(\widetilde{B}(n)) = 0$ if $n \not\equiv 2 \bmod 3$. The list of values of $\nu_2(\widetilde{B}(3n+2))$ is

$$\{1, 1, 2, 5, 1, 1, 2, 5, 1, 1, 2, 7, 1, 1, 2, 6, 1, 1, 2, 5, 1, 1, 2, 5, 1, 1, 2, 6, 1, 1\}$$

and the patterns $\{1, 1, 2, *\}$ suggest to consider the sequence $\nu_2(\widetilde{B}(n))$ for n modulo 12. The values $n \equiv 2 \bmod 3$ split into classes 2, 5, 8 and 11 modulo 12. The data suggests

$$\nu_2(\widetilde{B}(12n+5)) = 1,\ \nu_2(\widetilde{B}(12n+8)) = 1,\ \nu_2(\widetilde{B}(12n+11)) = 2,$$

while the class $n \equiv 2 \bmod 12$ does not exhibit such a pattern.

The first step in the analysis of 2-adic valuations of $\widetilde{B}(n)$ is to present some elementary congruences to establish that both $\widetilde{B}(3n)$ and $\widetilde{B}(3n+1)$ are always odd integers. The proof relies on the recurrence

$$\widetilde{B}(n) = -\sum_{k=0}^{n-1} \binom{n-1}{k} \widetilde{B}(k), \quad \text{for } n \geq 1 \text{ and } \widetilde{B}(0) = 1. \tag{6.2}$$

Proposition 6.1. *The complementary Bell numbers $\widetilde{B}(n)$ satisfy*

$$\widetilde{B}(3n) \equiv \widetilde{B}(3n+1) \equiv 1, \text{ and } \widetilde{B}(3n+2) \equiv 0 \bmod 2. \tag{6.3}$$

Proof. Proceed by induction. The recurrence (6.2) yields

$$-\widetilde{B}(3n) = \sum_{k=0}^{3n-1} \binom{3n-1}{k} \widetilde{B}(k). \tag{6.4}$$

Splitting the sum as

$$-\widetilde{B}(3n) = \sum_{k=0}^{n-1} \binom{3n-1}{3k} \widetilde{B}(3k) + \sum_{k=0}^{n-1} \binom{3n-1}{3k+1} \widetilde{B}(3k+1) + \sum_{k=0}^{n-1} \binom{3n-1}{3k+2} \widetilde{B}(3k+2)$$

and using the inductive hypothesis gives

$$-\widetilde{B}(3n) \equiv \sum_{k=0}^{n-1} \binom{3n-1}{3k} + \sum_{k=0}^{n-1} \binom{3n-1}{3k+1} \bmod 2. \tag{6.5}$$

The two sums appearing in the previous line add up to

$$2^{3n-1} - \sum_{k=0}^{n-1} \binom{3n-1}{3k+2}. \tag{6.6}$$

The result now follows from the identity

$$\sum_{k=0}^{n-1} \binom{3n-1}{3k+2} = \frac{2^{3n-1} + (-1)^n}{3}. \tag{6.7}$$

Both sides satisfies the recurrence $x_{n+2} - 7x_{n+1} - 8x_n = 0$ and have the same initial conditions $x_1 = 1$ and $x_2 = 11$. □

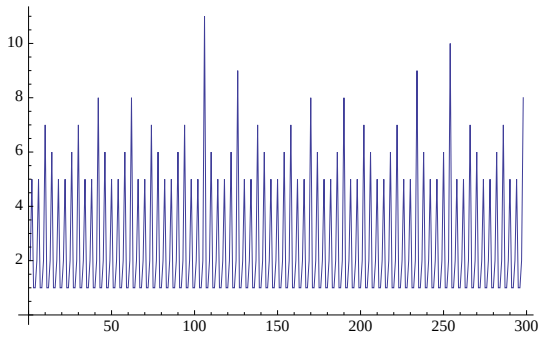

FIGURE 4. The 2-adic valuation of $\widetilde{B}(3n+2)$

Proposition 6.1 shows that

$$\nu_2(\widetilde{B}(3n)) = \nu_2(\widetilde{B}(3n+1)) = 0, \tag{6.8}$$

leaving the case $\nu_2(\widetilde{B}(3n+2))$ for discussion. This is presented in Section 7. Figure 4 shows the data for this sequence and its erratic behavior can be seen from the graph.

7. The 2-adic valuation of $\widetilde{B}(3n+2)$

The results from the previous section show that $\widetilde{B}(3n)$ and $\widetilde{B}(3n+1)$ are odd integers and $\widetilde{B}(3n+2)$ is an even integer. This section explores the value of the sequence $\nu_2(\widetilde{B}(3n+2))$. The family of polynomials $\{\lambda_j(k) : j \geq 0\}$ play the same role as $\mu_j(k)$ did for the regular Bell numbers $B(n)$.

Lemma 7.1. *The family of polynomials $\lambda_j(k)$, defined by*

$$\begin{aligned} \lambda_{j+1}(k) &= k\lambda_j(k) - \lambda_j(k+1), \\ \lambda_0(k) &= 1, \end{aligned} \tag{7.1}$$

satisfy

$$\widetilde{B}(n+j) = \sum_{k=0}^{n+j} (-1)^k S(n+j,k) = \sum_{k=0}^{n} (-1)^k \lambda_j(k) S(n,k), \tag{7.2}$$

for all n, $j \geq 0$.

Proof. Use the recurrence (7.1) and proceed as in the proof of Lemma 2.1. □

Corollary 7.2. *The evaluation $\widetilde{B}(j) = \lambda_j(0)$ is valid for $j \in \mathbb{N}$.*

The recursions for the descending factorials, given in Proposition 4.2 yield an evaluation of $\widetilde{B}(n)$ in terms of the powers of an infinite matrix.

Note. The (i,j)-entry of a matrix A is denoted by $A(i,j)$. This notation is used to prevent confusion with the presence of a variety of subindices.

Theorem 7.3. *Let $P = P(r,s)$, $r,s \geq 0$ be the infinite matrix defined by*

$$P(r+1,r) = 1, P(r,r) = r-1, P(r,r+1) = -r-1, \;\; P(r,s) = 0 \text{ for } |r-s| > 1 \tag{7.3}$$

or

$$P = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -2 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & -3 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 2 & -4 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 3 & -5 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{7.4}$$

Then

$$\widetilde{B}(n) = P^n(0,0). \tag{7.5}$$

Proof. The first step is to express the polynomials $\lambda_n(x)$ in terms of the falling factorial:

$$\lambda_n(k) = \sum_{r=0}^{n} c_n(r)(k)_r. \tag{7.6}$$

The recurrence relation in Lemma 7.1 shows that $c_n(r)$ are integers with $c_0(0) = 1$, $c_0(r) = 0$ for $r > 0$ and $c_n(r) = 0$ if $r > n$. Moreover, this recurrence may be expressed as

$$\mathbf{c_{n+1}} = P\mathbf{c_n}, \tag{7.7}$$

with P defined in (7.4) and $\mathbf{c_n}$ is the vector $(c_n(r) : r \geq 0)$.

Note that powers of P can be computed with a finite number of operations: each row or column has only finitely many non-zero entries. Iterating (7.7) gives

$$c_n(r) = P^n(r, 0), r \geq 0. \tag{7.8}$$

The result now follows from Corollary 7.2 and $c_n(0) = \lambda_n(0)$. □

The next lemma contains a precise description of the fact that the descending factorial $(k)_r$ is divisible by a large power of 2. This is a fundamental tool in the analysis of the 2-adic valuation of $\widetilde{B}(n)$.

Lemma 7.4. *For each $m \geq 0$ and $k \geq 1$, the congruence*

$$(k)_r \equiv 0 \mod 2^{2^m-1} \text{ holds for all } r \geq 2^m. \tag{7.9}$$

Proof. Since $(k)_j$ divides $(k)_j$ for $j \geq r$, it may be assumed that $r = 2^m$. Now observe that $(k)_r/r! = \binom{k}{r}$, thus $\nu_2((k)_r) \geq \nu_2(r!)$. For $r = 2^m$, Legendre's formula (1.5) gives the value $\nu_2(r!) = 2^m - s_2(2^m) = 2^m - 1$. □

Now we exploit the previous lemma to derive congruences for $\widetilde{B}(n)$ modulo a large power of 2. The first step is to show a result analogous to Theorem 7.3, with P replaced by a $2^m \times 2^m$ matrix, provided the computations are conducted modulo 2^{2^m-1}. Proposition 7.5 is not necessary for the results that follow it, but it is of interest because it allows us to express $\widetilde{B}(n)$ as the top left entry of the power of a finite matrix (with size depending on n).

Proposition 7.5. *Let $P[n]$ be the $n \times n$ matrix defined by*

$$P[n](r, s) = P(r, s), \quad 0 \leq r, s \leq n - 1. \tag{7.10}$$

For each $n \geq 1$ and $i \geq 1$,

$$\left(P[n]\right)^i (r, s) = P^i(r, s) \text{ for } 0 \leq r, s \leq n - 1, \quad r + s + i \leq 2n - 1.$$

Proof. Fix $n \geq 1$ and proceed by induction on i. The statement is clearly true for $i = 1$. Assume that $r + s + i + 1 \leq 2n - 1$, then the claim follows by computing

$$\left(P[n]\right)^{i+1} (r, s) = \sum_{t=0}^{n-1} \left(P[n]\right)^i (r, t) P[n](t, s). \tag{7.11}$$

□

Corollary 7.6. *For $i \leq 2n - 1$, the complementary Bell number is given by*

$$\widetilde{B}(i) = \left(P[n]\right)^i. \tag{7.12}$$

For $m \geq 1$ fixed, denote $P[2^m]$ by P_m. This is a matrix of size $2^m \times 2^m$, indexed by $\{0, 1, \ldots, 2^m - 1\}$. Lemma 7.4 gives

$$\lambda_n(k) \equiv \sum_{r=0}^{2^m-1} c_n(r)(k)_r \bmod 2^{2^m-1}, \quad n \geq 1, k \geq 0, \tag{7.13}$$

and then the same argument as before gives

$$c_n(r) \equiv P_m^n(r, 0) \mod 2^{2^m-1}, \text{ for } 0 \leq r \leq 2^m - 1, n \geq 1. \tag{7.14}$$

The next proposition summarizes the discussion.

Proposition 7.7. *For $n \in \mathbb{N}$,*

$$\widetilde{B}(n) \equiv P_m^n(0,0) \mod 2^{2^m-1}. \tag{7.15}$$

Corollary 7.8. *The complementary Bell numbers satisfy*

$$\widetilde{B}(n+j) \equiv \sum_{r=0}^{2^m-1} P_m^j(0,r) P_m^n(r,0) \mod 2^{2^m-1}, n \geq 1, j \geq 0. \tag{7.16}$$

Proof. This is simply the identity $P_m^{n+j} = P_m^n \times P_m^j$. □

Proposition 7.9. *The following table gives the values of $\widetilde{B}(24n+j)$ modulo 8 for $0 \leq j \leq 23$:*

j	$\widetilde{B}(24n+j) \mod 8$	j	$\widetilde{B}(24n+j) \mod 8$
0	1	12	5
1	7	13	3
2	0	14	0
3	1	15	5
4	1	16	5
5	6	17	6
6	7	18	3
7	7	19	3
8	2	20	2
9	3	21	7
10	5	22	1
11	4	23	4

Proof. Choose $m = 2$, and check that $P_2^{24} \equiv I \bmod 8$. Corollary 7.8 gives

$$\widetilde{B}(24n+j) \equiv \sum_{r=0}^{3} P_2^j(0,r) P_2^{24n}(r,0) \equiv P_2^j(0,0) \equiv \widetilde{B}(j) \mod 8. \tag{7.17}$$

Therefore the value of $\widetilde{B}(j)$ modulo 8 is a periodic function with period 24. The result follows by computing the values $\widetilde{B}(j)$ for $0 \leq j \leq 23$. □

Corollary 7.10. *Assume $j \not\equiv 2, 14 \bmod 24$. Then*

$$\nu_2(\widetilde{B}(j)) = \begin{cases} 1 & \text{if } j \equiv 5, 8, 17, 20 \bmod 24 \\ 2 & \text{if } j \equiv 11, 23 \bmod 24 \\ 0 & \text{otherwise.} \end{cases} \tag{7.18}$$

Corollary 7.11. *Assume $j \not\equiv 2, 14 \bmod 24$. Then $\widetilde{B}(j) \neq 0$.*

The remaining sections discuss the more difficult cases $n \equiv 2$ and $n \equiv 14 \bmod 24$.

8. The top-left block of powers of the matrix P_m

The analysis of the 2-adic valuation of $\widetilde{B}(n)$ employs the sequence of matrices appearing in the top-left block of powers of the matrix P_m. This section describes properties of this sequence.

A convention on their block structure is presented next:
let $n \in \mathbb{N}$ and i, j integers with $1 \leq i, j \leq n-1$. For an $n \times n$ matrix Q and an $i \times j$ matrix A, the block structure is

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{8.1}$$

Since the size of the top left corner determines the rest, the notation

$$Q = \begin{pmatrix} \overbrace{A}^{i\times j} & B \\ C & D \end{pmatrix}$$

will be used to specify the size of all blocks when necessary. The default convention is that whenever a $2^m \times 2^m$ matrix is written in block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, it will be understood that the blocks are of size $2^{m-1} \times 2^{m-1}$.

The next lemma is the essential part of the argument for the 2-adic analysis of $\widetilde{B}(n)$. The proof is a simple check with the definitions.

Definition 8.1. For each $m \geq 0$, define $2^m \times 2^m$ matrices B_m, D_m, V_m inductively as follows: $B_0 = -1, D_0 = 1, V_0 = 1$,

$$B_{m+1} = \begin{pmatrix} 0 & 0 \\ B_m & 0 \end{pmatrix}, \; D_{m+1} = \begin{pmatrix} D_m & B_m \\ 0 & D_m \end{pmatrix}, \; V_{m+1} = \begin{pmatrix} 0 & V_m \\ 0 & 0 \end{pmatrix},$$

where all blocks are $2^m \times 2^m$ matrices.

Recall the P_m is the $2^m \times 2^m$ matrix obtained from the top left corner of the infinite matrix P defined in (7.4).

Lemma 8.2. *The matrices P_m satisfy the recurrence*

$$P_{m+1} = \begin{pmatrix} P_m & 0 \\ V_m & P_m \end{pmatrix} + 2^m \begin{pmatrix} 0 & B_m \\ 0 & D_m \end{pmatrix}.$$

The first point in the analysis is to show that, for every power of P_m, the top half of the last column is zero modulo a large power of 2.

Lemma 8.3. *For all $m \geq 1$, $n \geq 1$, and $0 \leq i \leq 2^m - 1$, the inequality*

$$\nu_2\left(P_m^n(i, 2^m - 1)\right) \geq 2^m - m - 1 - \nu_2(i!). \tag{8.2}$$

holds.

Proof. The right-hand side vanishes for $m = 1$. Fix $m \geq 2$. If $n = 1$, the last column of P_m has $2^m - 2$ zeros at the beginning and its last two entries are $-(2^m - 1)$ and $2^m - 2$. Therefore, $\nu_2\left(P_m(i, 2^m - 1)\right) = \infty$ for $0 \leq i \leq 2^m - 3$, and

$$\begin{aligned} \nu_2\left(P_m(2^m - 2, 2^m - 1)\right) &= \nu_2(-(2^m - 1)) = 0, \\ \nu_2\left(P_m(2^m - 1, 2^m - 1)\right) &= \nu_2(2^m - 2) = 1. \end{aligned}$$

Legendre's formula (1.5) shows that the right-hand side of (8.2) is $2^m - m - 1 - i + s_2(i)$, so it vanishes for $i = 2^m - 2$ and $i = 2^m - 1$. This proves the case for $n = 1$.

The inductive step is presented next:

$$\begin{aligned} P_m^{n+1}(i, 2^m - 1) &= \sum_{j=0}^{2^m-1} P_m(i,j) P_m^n(j, 2^m - 1) \\ &= P_m(i, i-1) P_m^n(i-1, 2^m - 1) + P_m(i,i) P_m^n(i, 2^m - 1) \\ &\quad + P_m(i, i+1) P_m^n(i+1, 2^m - 1) \\ &= P_m^n(i-1, 2^m - 1) + (i-1) P_m^n(i, 2^m - 1) - (i+1) P_m^n(i+1, 2^m - 1). \end{aligned}$$

Observe that the three terms on the last line are elements of the last column of the matrix P_m^n. The inductive argument provides a lower bound on the power of 2 that divides these integers. Therefore, there are integers q_1, q_2, q_3 such that

$$P_m^{n+1}(i, 2^m - 1) = 2^{2^m - m - 1} \left(2^{-\nu_2((i-1)!)} q_1 + 2^{\nu_2(i-1) - \nu_2(i!)} q_2 - 2^{\nu_2(i+1) - \nu_2((i+1)!)} q_3 \right).$$

It follows that

$$\begin{aligned} &\nu_2 \left(P_m^{n+1}(i, 2^m - 1) \right) \geq \\ &\quad 2^m - m - 1 + \min\{ -\nu_2((i-1)!), \nu_2(i-1) - \nu_2(i!), \nu_2(i+1) - \nu_2((i+1)!) \}. \end{aligned} \tag{8.3}$$

Now use $\nu_2(i+1) - \nu_2((i+1)!) = -\nu_2(i!)$ and $-\nu_(i-1)!) \geq -\nu_2(i!)$, to verify that the minimum on the right is $-\nu_2(i!)$. This completes the argument. □

The next step is to describe the relation of the matrix P_m (of size $2^m \times 2^m$) to P_{m+1} (of size $2^{m+1} \times 2^{m+1}$). The additional block matrices appearing in this transition are defined recursively:

Fix $m \geq 0$, define $2^m \times 2^m$ matrices $V_{m,n}, A_{m,n}, B_{m,n}, C_{m,n}, D_{m,n}$ inductively by

$$V_{m,1} = V_m, \quad V_{m,n+1} = V_{m,n} P_m + P_m^m V_{m,n}$$

$$B_{m,1} = B_m, \quad B_{m,n+1} = P_m^n B_m + B_{m,n} P_m$$

$$A_{m,1} = 0, \quad A_{m,n+1} = A_{m,n} P_m + B_{m,n} V_m$$

$$D_{m,1} = D_m, \quad D_{m,n+1} = V_{m,n} B_m + P_m^n D_m + D_{m,n} P_m$$

$$C_{m,1} = 0, \quad C_{m,n+1} = C_{m,n} P_m + D_{m,n} V_m$$

The relation bewteen P_m and P_{m+1} is stated next.

Lemma 8.4. *For each $n \geq 1$, the congruence*

$$P_{m+1}^n \equiv \begin{pmatrix} P_m^n & 0 \\ V_{m,n} & P_m^n \end{pmatrix} + 2^m \begin{pmatrix} A_{m,n} & B_{m,n} \\ C_{m,n} & D_{m,n} \end{pmatrix} \mod 2^{2m} \tag{8.4}$$

holds.

Proof. The result is clear for $n = 1$. Computing $P_{m+1}^{n+1} = P_{m+1}^n P_{m+1}$, it follows that

$$\begin{aligned} P_{m+1}^{n+1} &\equiv \begin{pmatrix} P_m^n + 2^m A_{m,n} & 2^m B_{m,n} \\ V_{m,n} + 2^m C_{m,n} & P_m^n + 2^m D_{m,n} \end{pmatrix} \begin{pmatrix} P_m^n & 2^m B_m \\ V_{m,n} & P_m^n + 2^m D_m \end{pmatrix} \\ &\equiv \begin{pmatrix} P_m^{n+1} & 0 \\ V_{m,n}P_m + P_m^n V_m & P_m^{n+1} \end{pmatrix} \\ &+ 2^m \begin{pmatrix} A_{m,n}P_m + B_{m,n}V_m & P_m^n B_m + B_{m,n}P_m \\ C_{m,n}P_m + D_{m,n}V_m & V_{m,n}B_m + P_m^n D_m + D_{m,n}P_m \end{pmatrix} \mod 2^{2m}. \end{aligned}$$

The recurrence for the matrices A, B, C, D and V are designed to complete the inductive step. □

Corollary 8.5.

$$V_{m,2n} \equiv V_{m,n}P_m^n + P_m^n V_{m,n} \mod 2^{2m} \tag{8.5}$$

Proof. This follows from Lemma 8.4 by computing $P_{m+1}^{2n} = P_{m+1}^n P_{m+1}^n$. □

The next lemma shows some operational rules for the matrices A, B introduced above. The symbol $*$ indicates an unspecified integer or matrix.

Lemma 8.6. (a) *For any $2^m \times 2^m$ matrix $M(i,j)$ and arbitrary $i \in \mathbb{N}$, we have*

$$(MB_m)(i,0) = -M(i, 2^m - 1).$$

(b) *For $m \geq 2$ and $n \geq 1$, both $B_{m,n}$ and $A_{m,n}$ have the form*

$$\begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} \mod 2^{2^{m-1}-1}$$

Proof. Part (a) follows directly from the definition of B_m. Part (b) is established by induction. The statement holds for $B_{m,1}$. Now observe that

$$(P_m^n B_m)(i,0) = -P_m^n(i, 2^m - 1) \equiv 0 \mod 2^{2^{m-1}-1} \text{ for } 0 \leq i \leq 2^{m-1} - 1,$$

by part (a) and Lemma 8.3. The induction hypothesis implies that

$$B_{m,n} \equiv \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} \mod 2^{2^{m-1}-1},$$

and this leads to

$$B_{m,n+1} = P_m^n B_m + B_{m,n}P_m \equiv \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} \mod 2^{2^{m-1}-1}.$$

A similar argument shows that

$$A_{m,n+1} = A_{m,n}P_m + B_{m,n}V_m \equiv \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} \mod 2^{2^{m-1}-1}.$$

□

The next results describe the powers of P_m considered modulo 2^i. This leads to explicit formula for the 2-adic valuation of $\widetilde{B}(n)$.

Notation: $d_m = 3 \times 2^m$.

Proposition 8.7. *For all $m \geq 1$,*

$$P_m^{d_m} \equiv I \mod 4, \quad \textit{and} \quad V_{m,d_m} \equiv 0 \mod 2.$$

Proof. For $m = 1$, a direct calculation shows that $P_1^3 = I$ and so $P_1^{d_1} = P_1^6 = I$. Also,

$$V_{1,2} \equiv V_1P_1 + P_1V_1 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mod 2,$$

$$V_{1,3} \equiv V_{1,2}P_1 + P_1^2V_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \mod 2,$$

and this produces

$$V_{1,d_1} = V_{1,6} \equiv V_{1,3}P_1^3 + P_1^3V_{1,3} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mod 2.$$

Assume now $P_m^{d_m} \equiv I \bmod 4$ and $V_{m,d_m} \equiv 0 \bmod 2$. For simplicity, drop the subscripts in the matrices. Lemma 8.4 gives

$$P_{m+1}^{d_m} \equiv \begin{pmatrix} P & 0 \\ V & P \end{pmatrix} \equiv \begin{pmatrix} I & 0 \\ V & I \end{pmatrix} \bmod 4$$

and

$$P_{m+1}^{d_{m+1}} = \left(P_{m+1}^{d_m}\right)^2 \equiv \begin{pmatrix} I & 0 \\ V & I \end{pmatrix}\begin{pmatrix} I & 0 \\ V & I \end{pmatrix} \equiv \begin{pmatrix} I & 0 \\ 2V & I \end{pmatrix} \equiv \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \mod 4.$$

Using the notation

$$V_{m+1,d_m} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

it follows that

$$\begin{aligned} V_{m+1,d_{m+1}} &= V_{m+1,2d_m} \equiv V_{m+1,d_m}P_{m+1}^{d_m} + P_{m+1}^{d_m}V_{m+1,d_m} \\ &\equiv \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\begin{pmatrix} P & 0 \\ V & P \end{pmatrix} + \begin{pmatrix} P & 0 \\ V & P \end{pmatrix}\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \\ &\equiv \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\begin{pmatrix} I & 0 \\ V & I \end{pmatrix} + \begin{pmatrix} I & 0 \\ V & I \end{pmatrix}\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \\ &\equiv \begin{pmatrix} X+YV & Y \\ Z+WV & W \end{pmatrix} + \begin{pmatrix} X & Y \\ VX+Z & VY+W \end{pmatrix} \\ &\equiv \begin{pmatrix} 2X+YV & 2Y \\ 2Z+WV+VX & VY+2W \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mod 2. \end{aligned}$$

□

The next proposition provides the structure of $P_m^{d_m}$ module 2^{m+3}, for $m \geq 4$. Introduce the notation

$$Q = \begin{pmatrix} 1 & 2 & 6 & 0 \\ 6 & 1 & 0 & 6 \\ 3 & 4 & 5 & 4 \\ 0 & 1 & 4 & 3 \end{pmatrix}$$

and define recursively for $m \geq 4$ the $4 \times (2^m - 4)$ matrices R_m by

$$\begin{aligned} R_4 &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ R_{m+1} &= \begin{pmatrix} R_m & 0 \end{pmatrix}. \end{aligned}$$

Notation: $q(*)$ indicates a matrix or number that is a multiple of q.

Proposition 8.8. *Let $m \geq 4$. Then*

$$P_m^{d_m} \equiv I + \begin{pmatrix} \overbrace{2^m Q}^{4\times 4} & 2^{m+2}R_m \\ 4(*) & 4(*) \end{pmatrix} \mod 2^{m+3}.$$

Proof. The claim holds for $m = 4$ by *simple task*: evaluate P_4^{48} modulo 2^7. Keep in mind that P_4 is a 16×16 matrix.

Assume the claim holds for m. Observe that $2m \geq m+4$ for $m \geq 4$, therefore the congruence module 2^{2m} of Lemma 8.4 can be replaced with a congruence module 2^{m+4}. Write $V = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ to obtain

$$\begin{aligned} P_{m+1}^{d_m} &\equiv \begin{pmatrix} P & 0 \\ V & P \end{pmatrix} + 2^m \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &\equiv \begin{pmatrix} I + 2^m Q & 2^{m+2}R & 0 & 0 \\ 4(*) & I + 4(*) & 2^m(*) & 2^m(*) \\ X + 2^m(*) & Y + 2^m(*) & I + 2^m(*) & 2^m(*) \\ Z + 2^m(*) & W + 2^m(*) & 4(*) & I + 4(*) \end{pmatrix} \mod 2^{m+4}. \end{aligned}$$

Squaring this matrix gives

$$P_{m+1}^{d_{m+1}} \equiv \begin{pmatrix} I + 2^{m+1}Q & 2^{m+3}R & 0 & 0 \\ 4(*) & I + 4(*) & 4(*) & 4(*) \\ 2X + 4(*) & 2Y + 4(*) & I + 4(*) & 4(*) \\ 2Z + 4(*) & 2W + 4(*) & 4(*) & I + 4(*) \end{pmatrix} \mod 2^{m+4}.$$

The previous proposition shows that $V = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \equiv 0 \mod 2$, therefore

$$P_{m+1}^{d_{m+1}} \equiv I + \begin{pmatrix} 2^{m+1}Q & 2^{m+3}R_{m+1} \\ 4(*) & 4(*) \end{pmatrix} \mod 2^{m+4}.$$

This completes the induction argument. □

The next corollary is employed in the next section to establish the 2-adic valuation of complementary Bell numbers.

Corollary 8.9. *For each $n \geq 1$,*

$$P_m^{nd_m} \equiv I + n \begin{pmatrix} \overbrace{2^m Q}^{4\times 4} & 2^{m+2}R_m \\ 4(*) & 4(*) \end{pmatrix} \mod 2^{m+3}.$$

Proof. The result follows immediately from Proposition 8.8 and the binomial theorem. □

9. The case $n \equiv 2 \bmod 24$

The 2-adic valuations for the complementary Bell numbers $\widetilde{B}(n)$ are given in Corollary 7.10 for $j \not\equiv 2, 14 \bmod 24$. This section determines the case $j \equiv 2$.

The main result is :

Theorem 9.1. *For $n \in \mathbb{N}$,*

$$\nu_2\left(\widetilde{B}(24n+2)\right) = 5 + \nu_2(n).$$

Proof. Write $n = 2^m q$ with q odd. Corollary 7.8 and Proposition 8.8 give

$$\begin{aligned}
\widetilde{B}(24n+2) &= \widetilde{B}(3\cdot 2^{m+3}q+2) \equiv \sum_{r=0}^{2^{m+3}-1} P_{m+3}^{qd_{m+3}}(0,r)P_{m+3}^2(r,0) \\
&\equiv P_{m+3}^{qd_{m+3}}(0,0)P_{m+3}^2(0,0) + P_{m+3}^{qd_{m+3}}(0,1)P_{m+3}^2(1,0) \\
&\quad + P_{m+3}^{qd_{m+3}}(0,2)P_{m+3}^2(2,0) \\
&\equiv (1+2^{m+3}q)(0) - q2^{m+4} + 6q2^{m+3} \\
&\equiv q2^{m+5} \equiv 2^{m+5} \mod 2^{m+6}.
\end{aligned}$$

The expression for the valuation $\nu_2\left(\widetilde{B}(24n+2)\right)$ follows immediately. □

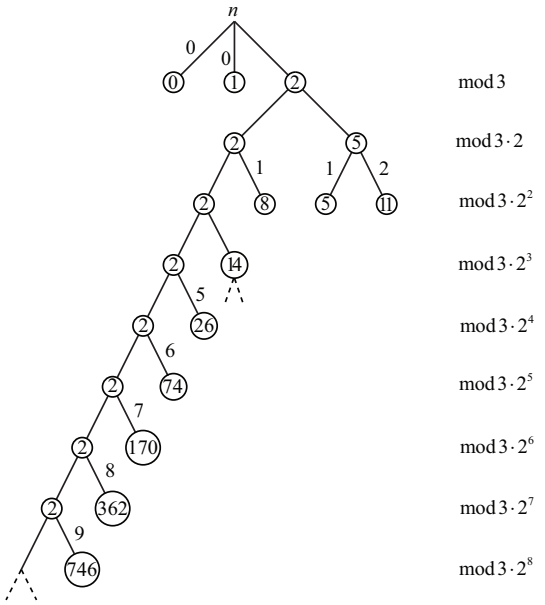

FIGURE 5. The 2-adic valuation of $\widetilde{B}(24n+2)$

The tree shown in Figure 5 summarizes the information derived so far on the 2-adic valuation of $\widetilde{B}(n)$. The top three edges of the tree correspond to the residue class of n (mod 3). The number by the side of the edge (if present) gives the (constant) 2-adic valuation of $\widetilde{B}(n)$ for that residue class. For example $\nu_2(\widetilde{B}(3n+1)) = 0$. If there is no number next to the edge, the 2-adic valuation is not constant for that residue class, so n needs to be split further. The split at each stage is conducted by replacing the index n of the sequence by $2n$ and $2n+1$. For example, the sequence $\nu_2(\widetilde{B}(12n+2))$ is not constant so it generates the two new sequences $\nu_2(\widetilde{B}(24n+2))$ and $\nu_2(\widetilde{B}(24n+14))$. Constant sequences include $\nu_2(\widetilde{B}(12n+8)) =$

$\nu_2(\widetilde{B}(12n+5)) = 1$ and $\nu_2(\widetilde{B}(12n+11)) = 2$. The main theorem of this section shows that the infinite branch on the left, coming from the splitting of $24n+2$, has a well-determined structure. The other infinite branch, corresponding to $24n+14$, does not exhibit such regular pattern. This is the topic of the next section.

10. The case $n \equiv 14 \pmod{24}$

This section discusses the last missing case in the 2-adic valuations of $\widetilde{B}(n)$. The main result of this section is:

Theorem 10.1. *There is at most one integer $n > 2$ such that $\widetilde{B}(n) = 0$.*

Outline of the proof. The proof consists of a sequence of steps.

Step 1. Define recursively two sequences $\{x_m,\, y_m\}$ via

$$\begin{aligned} y_{m+1} &= \begin{cases} y_m & \text{if } \nu_2(\widetilde{B}(x_m)) > m+5 \\ \\ y_m + 2^m & \text{if } \nu_2(\widetilde{B}(x_m)) \leq m+5 \end{cases} \\ \\ x_{m+1} &= 24y_{m+1} + 14. \end{aligned}$$

Step 2. Let $y_m = \displaystyle\sum_{i=0}^{m} s_{m,i}2^i$ and let $s_i = \lim_{m\to\infty} s_{m,i}$ and define $s = (s_0,\, s_1,\, s_2, \cdots)$.

Step 3. For $n \in \mathbb{N}$ let $n = \displaystyle\sum_k b_k(n)2^k$ be its binary expansion. Let

$$\omega(n) = \begin{cases} \text{first index } k \text{ such that } b_k(n) \neq s_k \\ \infty \qquad \text{otherwise} \end{cases} \tag{10.1}$$

Then $\omega(n) < \infty$ unless s has ony finitely many ones and s is the binary expansion of n. If such n exists, it is called *exceptional*.

Step 4. The 2-adic valuation of $\widetilde{B}(24n+14)$ is given by

$$\nu_2(\widetilde{B}(24n+14)) = \omega(n) + 5. \tag{10.2}$$

In particular $\widetilde{B}(n) = 0$ only if n is exceptional. This concludes the proof of the theorem.

Proof of Theorem 10.1.

The r-th entry of the top row of P_m^j needs to be expressed as a linear combination of $\widetilde{B}(j+i) \pmod{2^{2^m-1}}$, $0 \leq i \leq r$. This is the content of the next lemma.

Lemma 10.2. *Define $b_r(i)$ recursively by*

$$\begin{aligned} b_0(0) &= 1, \\ b_{r+1}(i) &= b_r(i-1) + (1-r)b_r(i) + rb_{r-1}(i), \quad 0 \leq i \leq r \\ b_r(i) &= 0 \quad \textit{for } i < 0 \textit{ or } i > r. \end{aligned}$$

Then for each $m \geq 1$, $j \geq 1$, and $0 \leq r \leq 2^m - 1$, we have

$$P_m^j(0,r) \equiv \sum_{i=0}^{r} b_r(i)\widetilde{B}(j+i) \pmod{2^{2^m-1}}.$$

Proof. The proof is by induction on r. If $r = 0$, the statement is Proposition 7.7. Assuming the statement for r, it follows that

$$P_m^{j+1}(0,r) \equiv \sum_{i=0}^{r} b_r(i)\widetilde{B}(j+1+i) \mod 2^{2^m-1}$$

and also

$$\begin{aligned} P_m^{j+1}(0,r) &= P_m^j(0,r-1)P_m(r-1,r) + P_m^j(0,r)P_m(r,r) \\ &+ P_m^j(0,r+1)P_m(r+1,r) \\ &= -rP_m^j(0,r-1) + (r-1)P_m^j(0,r) + P_m^j(0,r+1). \end{aligned}$$

Comparing the two expressions and using induction, $P_m^j(0,r+1)$ is expressed as a linear combination of $\widetilde{B}(j+i)$, $0 \le i \le r$, with coefficients as in the right side of the equation defining $b_{r+1}(i)$. □

Extensive calculations suggest that $\nu_2(\widetilde{B}(24n+14))$ is always at least 5, and it is rather irregular. After examining the experimental data, we were led to define the following sequences.

Define x_m, y_m inductively by:

$$y_0 = 0, \quad x_0 = 24y_0 + 14,$$

and if x_m, y_m have been defined, set

$$y_{m+1} = \begin{cases} y_m & \text{if } \nu_2\left(\widetilde{B}(x_m)\right) > m+5 \\ 2^m + y_m & \text{if } \nu_2\left(\widetilde{B}(x_m)\right) \le m+5 \end{cases}, \quad x_{m+1} = 24y_{m+1} + 14.$$

This is the statement of Step 1.

The next table gives the first few values of y_m and x_m.

m	0	1	2	3	4	5	6	7	8	9	10
y_m	0	1	1	5	13	13	13	77	77	333	845
x_m	14	38	38	134	326	326	326	1862	1862	8006	20294

The next lemma provides a lower bound for the 2-adic valuation of the subsequence of complementary Bell numbers indexed by x_m.

Lemma 10.3. *For $m \in \mathbb{N}$, $\nu_2(\widetilde{B}(x_m)) \ge m+5$.*

Proof. The proof employs the values of $b_r(i)$ for $0 \le r \le 2$. These are given in Lemma 10.2 for $r = 0, 1, 2$. It turns out that $b_1(0) = b_1(1) = b_2(0) = b_2(1) = b_2(2) = 1$. (In case one wonders here if all non-zero terms of $b_r(i)$ are 1, this is not true for $r \ge 3$).

Direct calculation shows that $\nu_2(\widetilde{B}(x_0)) = \nu_2(\widetilde{B}(14)) = 5$, and $\nu_2(\widetilde{B}(x_1)) = \nu_2(\widetilde{B}(38)) = 7$. Therefore the statement holds for $m = 0$, 1. Assume the result for $m \ge 1$. Therefore $\nu_2(\widetilde{B}(x_m)) \ge m+5$. If $\nu_2(\widetilde{B}(x_m)) > m+5$, then by definition $x_{m+1} = x_m$, and it follows that $\nu_2(\widetilde{B}(x_{m+1})) \ge m+6$. On the other hand, if $\nu_2(\widetilde{B}(x_m)) = m+5$, write $\widetilde{B}(x_m) = 2^{m+5}q$, with q is odd. Then $y_{m+1} = 2^m + y_m$, and $x_{m+1} = 24(2^m + y_m) + 14 = 3 \cdot 2^{m+3} + x_m$. Corollary 7.8 (with $n = 3 \cdot 2^{m+3}$, $j =$

x_m, and m replaced by $m+3$) and Proposition 8.8 (with m replaced by $m+3$), produce

$$\begin{aligned}\widetilde{B}(x_{m+1}) &= \widetilde{B}(3\cdot 2^{m+3}+x_m) \equiv \sum_{r=0}^{2^{m+3}-1} P_{m+3}^{x_m}(0,r)P_{m+3}^{d_{m+3}}(r,0) \mod 2^{2^{m+3}-1} \\ &\equiv (1+2^{m+3})P_{m+3}^{x_m}(0,0)+6\cdot 2^{m+3}P_{m+3}^{x_m}(0,1)+3\cdot 2^{m+3}P_{m+3}^{x_m}(0,2) \\ &\quad + \sum_{r=4}^{2^{m+3}-1} P_{m+3}^{x_m}(0,r)P_{m+3}^{d_{m+3}}(r,0) \mod 2^{m+6}.\end{aligned}$$

Proposition 8.8 shows that the first term in the last sum is divisible by 2^{m+5} and the second term is divisible by 4. Then, Lemma 10.2 yields

$$\begin{aligned}\widetilde{B}\,(x_{m+1}) &\equiv (1+2^{m+3})\widetilde{B}\,(x_m)+3\cdot 2^{m+4}\left(\widetilde{B}\,(x_m)+\widetilde{B}\,(x_m+1)\right) \\ &\quad +3\cdot 2^{m+3}\left(\widetilde{B}\,(x_m)+\widetilde{B}\,(x_m+1)+\widetilde{B}\,(x_m+2)\right) \mod 2^{m+6}.\end{aligned}$$

Since $x_m+1\equiv 15$ and $x_m+2\equiv 16 \bmod 24$, Proposition 7.9 shows that $\widetilde{B}\,(x_m+1)\equiv \widetilde{B}\,(x_m+2)\equiv 5 \pmod 8$. So we find

$$\begin{aligned}\widetilde{B}\,(x_{m+1}) &\equiv (1+2^{m+3})2^{m+5}q+3\cdot 2^{m+4}\left(2^{m+5}q+5+8(*)\right) \\ &+ 3\cdot 2^{m+3}\left(2^{m+5}q+5+8(*)+5+8(*)\right) \\ &\equiv 2^{m+5}q+15\cdot 2^{m+4}+15\cdot 2^{m+3}+15\cdot 2^{m+3} \\ &\equiv 2^{m+5}q+15\cdot 2^{m+5}\equiv (q+15)2^{m+5}\equiv 0 \mod 2^{m+6}.\end{aligned}$$

This completes the inductive step. □

Lemma 10.4. *The binary expansion of y_m has the form*

$$y_m=\sum_{i=0}^{m} s_{m,i}2^i \tag{10.3}$$

and $s_i=\lim_{m\to\infty} s_{m,i}$ exists.

Proof. By construction $y_m\le 2^m-1$, showing that the binary expansion of y_m ends at 2^{m-1}. Moreover, the binary expansion of y_{m+1} is the same as that of y_m with possibly and extra leading 1. This confirms the existence of the limit s_i. □

Note. Step 2 concludes by defining $s=(s_0,s_1,\ldots)=(1,0,1,1,0,0,1,0,1,1,\ldots)$.

Theorem 10.5. *Let n be a positive integer with binary expansion $n=\sum_k b_k2^k$, and let $\omega(n)$ be the first index for which $b_k\neq s_k$. If no such index exists, let $\omega(n)=\infty$. Then*

$$\nu_2(\widetilde{B}(24n+14))=\omega(n)+5.$$

Note. As discussed in Step 3, there is at most one index $n>2$ for which $\omega(n)=\infty$. This happens when s, defined above, has finitely many ones. In this situation, s is the binary expansion of this exceptional index. The conjecture of Wilf states that this situation *does not happen*.

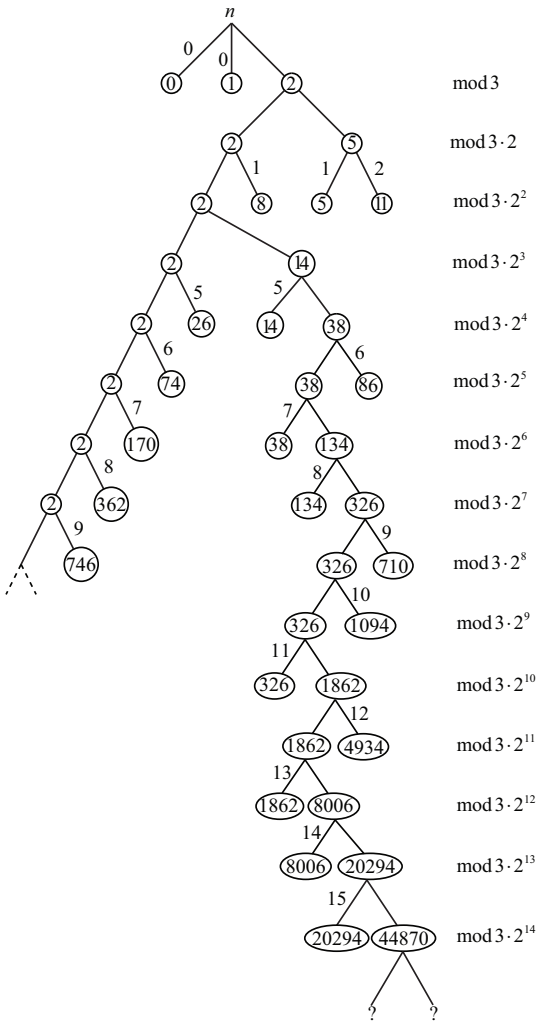

FIGURE 6. The 2-adic valuation of $\widetilde{B}(24n+14)$

Proof. The notation $m = \omega(n)$ is employed in the proof. If $m = \infty$, then $\widetilde{B}(24n + 14) = 0$ and the formula holds. Suppose now that $m \neq \infty$. Then there is $p \in \mathbb{N}$ such that $24n + 14 = 3 \cdot 2^{m+3}p + x_m$.

Write $\widetilde{B}(x_m) = 2^{m+5+i}q$, with q odd and $i \geq 0$. Then, as in the previous proof (and also using Lemma 8.9), it follows that

$$\begin{aligned}
\widetilde{B}(24n+14) &= \widetilde{B}\left(3 \cdot 2^{m+3}p + x_m\right) \\
&\equiv (1 + 2^{m+3}p)2^{m+5+i}q + 3p \cdot 2^{m+4}\left(2^{m+5+i}q + 5 + 8(*)\right) \\
&+ 3p \cdot 2^{m+3}\left(2^{m+5+i}q + 5 + 8(*) + 5 + 8(*)\right) \\
&\equiv 2^{m+5+i}q + 15p \cdot 2^{m+4} + 15p \cdot 2^{m+3} + 15p \cdot 2^{m+3} \\
&\equiv 2^{m+5+i}q + 15p \cdot 2^{m+5} \equiv 2^{m+5}(2^i q + 15p) \mod 2^{m+6}.
\end{aligned}$$

If $i = 0$, then $s_m = 1$, and p must be even (because this is where n and s disagree). Thus the quantity in parentheses on the last line is odd, and $\nu_2(\widetilde{B}(24n + 14)) = m + 5$. If $i > 0$, then $s_m = 0$, and p must be odd and, as in the previous case, the quantity in parentheses is odd. The result follows from here. □

Note. The tree shown in Figure 6 updates Figure 5 by including the 2-adic valuation of $\widetilde{B}(24n + 14)$. It is a curious fact that $\nu_2(\widetilde{B}(n))$ takes on all non-negative values except 3 and 4.

Final comment. It remains to decide if the exceptional case exists. If it does not, then $\widetilde{B}(n) \neq 0$ for $n > 2$, Wilf's conjecture is true and the sequence $\nu_2(\widetilde{B}(24n+14))$ is unbounded. If this exceptional index exists, then it is unique. Observe that the exceptional case exists if and only if the sequence x_m is eventually constant.

11. Two classes of polynomials

Two families of polynomials have been considered in Lemma 2.1 and Lemma 7.1: $\mu_0(x) \equiv 1, \lambda_0(x) \equiv 1$, and

$$\mu_{j+1}(x) = x\mu_j(x) + \mu_j(x+1); \quad \text{for } n \geq 0; \tag{11.1}$$

$$\lambda_{j+1}(x) = x\lambda_j(x) - \lambda_j(x+1); \quad \text{for } n \geq 0. \tag{11.2}$$

The corresponding exponential generating functions are provided below.

Lemma 11.1. *The polynomials μ_j and λ_j have generating functions given by*

$$\sum_{j=0}^{\infty} \frac{z^j}{j!}\mu_j(x) = e^{xz-1+e^z} \quad \text{and} \quad \sum_{j=0}^{\infty} \frac{z^j}{j!}\lambda_j(x) = e^{xz+1-e^z}. \tag{11.3}$$

Proof. Let $F(x,z) = \displaystyle\sum_{j\geq 0} \frac{z^j}{j!}\mu_j(x)$ and $G(x,z) = e^{xz-1+e^z}$. Multiplying the polynomial recurrence through by $z^j/j!$ yields

$$\mu_{j+1}(x)\frac{z^j}{j!} = x\mu_j(x)\frac{z^j}{j!} + \mu_j(x+1)\frac{j^n}{j!}.$$

Now sum over all non-negative integers j to find

$$\frac{\partial}{\partial z}F(x,z) = xF(x,z) + F(x+1,z). \tag{11.4}$$

Since $G(x+1,z) = e^z G(x,z)$, it follows

$$\frac{\partial}{\partial z}G(x,z) = G(x,z)(x+e^z) = xG(x,z) + G(x+1,z). \tag{11.5}$$

On the other hand, $F(x,0) = \mu_0(x) = 1 = G(x,0)$. Therefore, $F(x,z) = G(x,z)$. The same argument verifies the second assertion of the lemma. The proof is complete. □

Corollary 11.2. *The polynomials μ_j and λ_j satisfy*

$$\mu_j(0) = B(j) \text{ and } \lambda_j(0) = \widetilde{B}(j). \tag{11.6}$$

Corollary 11.3. *There are double-indexed exponential generating functions for* $\mu_j(n), \lambda_j(n)$*:*

$$\sum_{j,n\geq 0} \mu_j(n)\frac{z^j y^n}{j!n!} = e^{-1+(y+1)e^z}, \quad \sum_{j,n\geq 0} \lambda_j(n)\frac{z^j y^n}{j!n!} = e^{-1+(y-1)e^z}.$$

Proof. Direct computation shows

$$\sum_{j,n} \mu_j(n)\frac{z^j y^n}{j!n!} = \sum_n e^{nz-1+e^z}\frac{y^n}{n!} = e^{-1+e^z}\sum_n \frac{(ye^z)^n}{n!} \tag{11.7}$$

with a similar argument for λ_j. □

Corollary 11.4. *The polynomials* $\mu_j(x), \lambda_j(x)$ *are binomial convolutions of Bell numbers,*

$$\mu_j(x) = \sum_r \binom{j}{r} B(r)x^{j-r}, \quad \lambda_j(x) = \sum_r \binom{j}{r}\widetilde{B}(r)x^{j-r}.$$

Proof. This follows directly from

$$\sum_{j\geq 0} \mu_j(x)\frac{z^j}{j!} = e^{e^z-1}e^{xz} = \sum_{k\geq 0} B(k)\frac{z^k}{k!} \times \sum_{n\geq 0} x^n\frac{z^n}{n!} \tag{11.8}$$

and a similar argument for λ_j. □

Corollary 11.5. *The family of polynomials* $\lambda_j(x)$ *have a missing strip of coefficients, i.e.*

$$[x^{j-2}]\lambda_j(x) = 0.$$

Proof. Follows from Corollary 11.4 and $\widetilde{B}(2) = 0$. □

Define inductively the functions

$$\begin{aligned} e(x) &= e^{(1)}(x) = 1 - e^x \\ e^{(k+1)}(x) &= e(e^{(k)}(x)). \end{aligned}$$

These are called *super-exponentials*. For example,

$$e^{(2)}(x) = 1 - e^{1-e^x} \quad \text{and} \quad e^{(3)}(x) = 1 - e^{1-e^{1-e^x}}.$$

Introduce the *super-complementary Bell numbers*, $\widetilde{B}^{(k)}(n)$, according to

$$\sum_{n\geq 0} \widetilde{B}^{(k)}(n)\frac{x^n}{n!} = 1 - e^{(k+1)}(x). \tag{11.9}$$

The usual complementary Bell numbers $\widetilde{B}(n)$ become $\widetilde{B}^{(1)}(n)$ due to the relation

$$\sum_n \widetilde{B}(n)\frac{x^n}{n!} = e^{1-e^x} = 1 - e^{(2)}(x). \tag{11.10}$$

The next conjecture is a natural extension of Wilf's original question.

Conjecture 11.6. *Let* $k \in \mathbb{N}$ *be odd. Then* $\widetilde{B}^{(k)}(n) = 0$ *if and only if* $n = 2$*. For* $k \in \mathbb{N}$ *even and* $k \neq 2$*, it is conjectured that* $\widetilde{B}^{(k)}(n) \neq 0$*. The case* $k = 2$ *is peculiar: the corresponding conjecture is that* $\widetilde{B}^{(2)}(n) = 0$ *if and only if* $n = 3$*.*

Combinatorial meanings: $B_1^{(1)}(n)$ = number of set partitions of $\{1,\dots,n\}$ with an even number of parts, minus the number of such partitions with an odd number of parts; $B_1^{(2)}(n)$ = number of set partitions of $\{1,\dots,n\}$ with an even number of parts, minus the number of such partitions with an odd number of parts, and then repeating this process for each block. Similar number of chain reactions yield $B_1^{(k)}(n)$. For instance,

$$\widetilde{B}^{(2)}(n)=\sum_{j=0}^{n}(-1)^j S(n,j)\widetilde{B}(j). \tag{11.11}$$

Illustrative example. Take $n=3$, and partition the set $\{1,2,3\}$. For $k=1$: $\{1,2,3\}$; for $k=2$: $\{1,\{2,3\}\}$, $\{2,\{1,3\}\}$, $\{3,\{1,2\}\}$; for $k=3$: $\{\{1\},\{2\},\{3\}\}$. In the next step, partition blocks as follows. When $k=1$: $\{1,2,3\}$ is its own partition as a 1-element set; when $k=2$, partition each of $\{1,\{2,3\}\}$, $\{2,\{1,3\}\}$, $\{3,\{1,2\}\}$ as 2-element sets; when $k=3$, partition $\{\{1\},\{2\},\{3\}\}$ as a 3-element set. The resulting collection looks like this:

$\{1,2,3\},\{1,\{2,3\}\},\{\{1\},\{\{2,3\}\}\},\{2,\{1,3\}\},\{\{2\},\{\{1,3\}\}\},\{3,\{1,2\}\},$
$\{\{3\},\{\{1,2\}\}\},\{\{1\},\{2\},\{3\}\},\{\{1\},\{\{2\},\{3\}\}\},\{\{2\},\{\{1\},\{3\}\}\},\{\{3\},\{\{1\},\{2\}\}\},$
$\{\{1\},\{\{2\}\},\{\{3\}\}\}.$

Acknowledgements. The last author acknowledges the partial support of NSF-DMS 0713836.

References

[1] N. C. Alexander. Non-vanishing of Uppuluri-Carpenter. *Preprint*. 2006.

[2] T. Amdeberhan, D. Manna, and V. Moll. The 2-adic valuation of a sequence arising from a rational integral. *Jour. Comb. A*, 2008.

[3] T. Amdeberhan, D. Manna, and V. Moll. The 2-adic valuation of Stirling numbers. *Experimental Mathematics*, 17:69–82, 2008.

[4] T. Amdeberhan, L. Medina, and V. Moll. Arithmetical properties of a sequence arising from an arctangent sum. *Journal of Number Theory*, 128:1808–1847, 2008.

[5] J. An. Wilf conjecture. *Preprint*. 2008.

[6] G. P. Egorychev and E. V. Zima. Integral representation and algorithms for closed form summation. In *Handbook of Algebra*, volume 5, pages 459–529. Elsevier, 2008.

[7] M. Klazar. Bell numbers, their relatives and algebraic differential equations. *J. Comb. Theory Ser. A*, 102:63–87, 2003.

[8] M. Klazar. Counting even and odd partitions. *Amer. Math. Monthly*, 110:527–532, 2003.

[9] V. Moll and X. Sun. A binary tree representation for the 2-adic valuation of a sequence arising from a rational integral. *INTEGERS*, 10:211–222, 2010.

[10] M. Ram Murty and S. Sumner. On the p-adic series $\sum_{n=1}^{\infty} n^k \cdot n!$. In H. Kisilevsky and E. Z. Goren, editors, *CRM Proceedings and Lectures Notes. Number Theory, Canadian Number Theory Association VII*, pages 219–228. Amer. Math. Soc., 2004.

[11] M. Petkovsek, H. Wilf, and D. Zeilberger. *A=B*. A. K. Peters, Ltd., 1st edition, 1996.

[12] T. Laffey S. De Wannemacker and R. Osburn. On a conjecture of Wilf. *Journal of Combinatorial Theory Series A*, 114:1332–1349, 2007.

[13] M. V. Subbarao and A. Verma. Some remarks on a product expansion: An unexplored partition function. In F. G. Garvan and M. E. H. Ismail, editors, *Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics, Gainsville, Florida, 1999*, pages 267–283. Kluwer, Dordrecht, 2001.

[14] X. Sun and V. Moll. The p-adic valuation of sequences counting Alternating Sign Matrices. *Journal of Integer Sequences*, 12:09.3.8, 2009.

[15] V. R. Rao Uppuluri and J. A. Carpenter. Numbers generated by the function $\exp(1 - e^x)$. *Fib. Quart.*, 7:437–448, 1969.

[16] Y. Yang. On a multiplicative partition function. *Elec. Jour. Comb.*, 8:Research Paper 19, 2001.

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: tamdeberhan@math.tulane.edu

Department of Mathematics, Xavier University of Louisiana, New Orleans, LA 70125
E-mail address: vdeangel@xula.edu

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: vhm@math.tulane.edu