The Classical Proof of the Prime Number Theorem

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December 30, 2023

Abstract

The Prime Number Theorem states that the number of prime numbers less than x is asymptotic to $x/\log x$ as x becomes large. Its proof was a crowning achievement of 19th century mathematics, and led to the development of a vast amount of mathematics ever since. So understanding the classical proof is more than just of historical interest. This article details a minimal path to that first classical proof.

1 Introduction

The first complete proof of the Prime Number Theorem was given (independently) by Hadamard and de la Vallé Poussin in 1896 [4], [1]. It was the culmination of work by many of the leading mathematicians of the 18th and 19th centuries such as Euler, Chebyshev, Riemann and von Mangoldt.

Many simpler and "elementary" proofs of the theorem have been produced in the following 128 years [3, 8, 5, 9]. However, the ideas involved in the classical proof have been responsible for the development of a large amount of 20th and 21st century mathematics. In particular, Riemann's short and legendary 1859 paper [7] can be rightly considered to be the birth of Analytic Number Theory.

This article describes the first complete proof of the theorem. H. Edwards's masterly historical account [2] traces (in the first four chapters) the development of the proof from Riemann's 1859 paper until the final steps by Hadamard and de la Vallé Poussin in 1896, and it provides the main reference work for this article. While Edwards's account is focussed on the theory of the Riemann zeta function, the present article aims at providing a minimal path to the proof of the Prime Number Theorem.

Section 2 consists of five theorems and six propositions that summarize the important steps needed in the proof. No proofs of the results are given, but the logical dependence of the various steps is outlined.

Section 3 provides full proofs of Theorems 1,2 and 5. For Theorem 3 (the product expansion for $\xi(s)$) and Theorem 4 (the analytic formula for $\psi(x)$ and its antiderivative), an outline is given, consisting of several lemmas, just like Section 2 provides an outline for the Prime Number Theorem.

Section 4 provides full proofs of the six propositions. Section 5 provides the proofs of the lemmas needed for the proofs of Theorems 3 and 4. Notes in the appendix provide background material and proofs of standard facts quoted in the proofs. A web based version of this article is available at https://vdeangel.xula.edu/PNT.html

2 Outline

2.1 Statement of the Prime Number Theorem

Let $\pi(x)$ be the prime counting function

$$\pi(x) = |\{p : p \text{ is prime and } p \leq x\}|.$$

For positive functions $f(x)$ and $g(x)$, the notation $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$ as $x \to \infty$. The Prime Number Theorem states that

$$\pi(x) \sim \mathrm{Li}(x),$$

where

$$\mathrm{Li}(x) = \lim_{\epsilon \to 0^+} \left(\int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^{x} \frac{dt}{\log t} \right).$$

It is easy to see that $\mathrm{Li}(x) \sim x/\log x$, so an equivalent statement of the Prime Number Theorem is $\pi(x) \sim x/\log x$.

2.2 Chebyshev's function

The prime counting function $\pi(x)$ arises from the point measure $d\pi(t)$ that assigns weight 1 to prime numbers and 0 to every other number. Let $d\theta(t) = (\log t) d\pi(t)$. The corresponding function

$$\theta(x) = \sum_{p<x} \log p$$

(where the sum is over the prime numbers) is *Chebyshev's function.* We assume by convention that the values of step functions such as this at the points of discontinuity are defined to be the average of their left and right limits. The next proposition shows that in order to prove the PNT, it is enough to derive the asymptotic behavior of $\theta(x)$.

Proposition 1.

$$\theta(x) \sim x \textit{ implies } \pi(x) \sim \mathrm{Li(x)}.$$

2.3 von Mangoldt's function

The function $\theta(x)$ is constructed using prime numbers only. If we also include prime powers, we get von Mangoldt's function

$$\psi(x) = \sum_{p^n < x} \log p = \sum_{n<x} \Lambda(n),$$

where the first sum is over all prime powers p^n, and

$$\Lambda(n) = \begin{cases} \log p & \text{if } n \text{ is a prime power } p^m \\ 0 & \text{if } n \text{ is not a prime power} \end{cases}.$$

von Mangoldt's function has a very useful connection with the infinite sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \ \mathrm{Re}(s) > 1. \tag{1}$$

The connection is via Euler's product formula [A1]

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1,$$

that is the heart of the Prime Number Theorem. Let $d\psi(t)$ be the point measure corresponding to $\psi(t)$, that assigns weight $\log p$ to prime powers p^n, and zero to all other numbers.

Proposition 2.

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_0^\infty \frac{d\psi(x)}{x^s} = \sum_{n=2}^{\infty} \frac{1}{n^s}\Lambda(n), \quad Re(s) > 1 \tag{2}$$

Equation (2) can be considered a re-statement in Stieltjes integral form of Euler's product formula.

2.4 The PNT from $\psi(x) \sim x$

The heart of the proof of the Prime Number Theorem consists in exploiting equation (2) to derive the asymptotic estimates

$$\psi(x) \sim x.$$

Once that is done, the Prime Number Theorem is a consequence of the following simple result.

Proposition 3.

$$\psi(x) \sim x \textit{ implies } \theta(x) \sim x.$$

2.5 Extension of $\zeta(s)$

So we have now reduced the proof of the PNT to the derivation of the asymptotic behavior of $\psi(x)$. This is where complex analysis comes in. The function $\zeta(s)$ in (1) is Riemann's zeta function, that had been used by Euler mostly for integer values of s, and later by Dirichlet for real values $s > 1$.

The crucial connection (2) between the prime numbers and $\zeta(s)$ can now be exploited using Riemann's new and far reaching idea that $\zeta(s)$ extends analytically to a meromorphic function with just a simple pole at $s = 1$, and then using complex function theory to derive results on the prime numbers. Riemann's work in this area can be considered the birth of Analytic Number Theory.

In order to state the next theorem, we introduce the extension of the factorial [Appendix B]

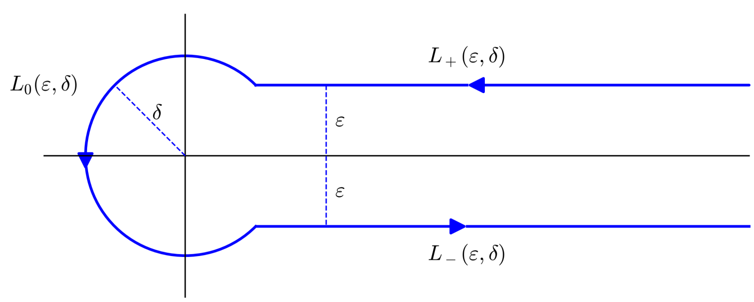
$$\Pi(s) = \prod_{k=1}^{\infty} \left(1 + \frac{s}{k}\right)^{-1} \left(1 + \frac{1}{k}\right)^{s} \tag{3}$$

that was already known to Euler. $\Pi(s)$ is defined and analytic for all s except the negative integers, and $\Pi(n) = n!$ if n is a non-negative integer. A more common (but less natural) notation for it is $\Gamma(s) = \Pi(s-1)$.

Theorem 1. *For each $s \neq 1$, define*

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i} \int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z}, \tag{4}$$

where $C = C(\delta, \varepsilon)$ is the Hankel path that starts at a small distance ε above the positive real axis at infinity, circles counterclockwise around the origin with radius $\delta > \varepsilon$, and returns to infinity traveling at distance ε below the positive real axis (see picture below).



The Hankel path

Then $\zeta(s)$ is a meromorphic function with a simple pole at $s = 1$, and it coincides with the previous definition for $Re(s) > 1$. Moreover, $\zeta(-2n) = 0$ for all positive integers n.

The first few values of $\zeta(s)$ are:

n	0	-1	-2	-3	-4	-5
$\zeta(n)$	$-\frac{1}{2}$	$-\frac{1}{12}$	0	$\frac{1}{120}$	0	$-\frac{1}{252}$

The zeros at $s = -2n$ are called the *trivial* zeros. But $\zeta(s)$ has many other zeros, which we will denote by ρ, that are crucial in understanding the distribution of prime numbers. The Riemann hypothesis, proposed in his 1859 paper, is that all the non-trivial zeros ρ of $\zeta(s)$ have $\mathrm{Re}(\rho) = 1/2$.

2.6 The function $\xi(s)$

So the zeta function contains information about the prime numbers. But in order to extract it in the most efficient way, we now strip $\zeta(s)$ of some inessential information. First of all we multiply it by $s-1$ in order to make it an entire function. Then we multiply it by $\Pi(s/2)$ in order to eliminate the trivial zeros at the negative even integers. At this point we would be left with an entire function without real zeros (because as the next theorem shows $\mathrm{Im}(\rho) \neq 0$ for all ρ). An additional simple factor of $\pi^{-s/2}$ will turn it into the function that plays a central role in the proof of the PNT, and that is described in the next theorem.

Theorem 2. *Let*

$$\xi(s) = (s-1)\Pi\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s).$$

Then:

(a) $\xi(s)$ is entire.

(b) $\xi(s)$ satisfies the functional equation

$$\xi(1-s) = \xi(s). \tag{5}$$

(c) $\xi(s)$ has no zeros outside the strip $0 \leq Re(s) \leq 1$.

(d) $\xi(s)$ is an even function of $s - 1/2$, and the coefficients of the Taylor expansion of $\xi(s)$ at $s = 1/2$ are positive real numbers. In particular, $\xi(s)$ has no zeros on the real axis.

Note that as a consequence of the functional equation (5), the zeros of $\xi(s)$ will occur in pairs $(\rho, 1-\rho)$.

2.7 The product expansion for $\xi(s)$

Riemann had the insight that the function $\xi(s)$ admits an infinite product expansion whose factors are $1 - s/\rho$, where ρ are the zeros of $\xi(s)$, and that this infinite product expansion is the key to the derivation of the information about the prime numbers.

This product expansion (that essentially says that $\xi(s)$ behaves like an infinite degree polynomial) is similar to Euler's expansion for $\sin(\pi s)$ in terms of factors $1 - s^2/n^2$. But Riemann did not prove his statement in his short 1859 paper. He simply stated that the Taylor series for $\xi(s)$ around $s = 1/2$ converges "very rapidly". A rigorous proof of the product expansion for $\xi(s)$ (described in the next theorem) was given by Hadamard about 30 years later, and it was the first major progress towards a rigorous proof of the PNT following Riemann's 1859 paper. The proof of this theorem occupies a whole chapter in Edwards' book.

Theorem 3. *The function $\xi(s)$ has a product expansion*

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right),$$

where the product is over all the zeros ρ of $\xi(s)$, and the zeros $\rho, 1-\rho$ are assumed to be paired, so that the product is

$$\xi(s) = \xi(0) \prod_{\mathrm{Im}(\rho)>0} \left(1 - \frac{s(1-s)}{\rho(1-\rho)}\right).$$

2.8 The integral $I(b)$

The next important step was the derivation of an analytic formula for $\psi(x)$. This was achieved by von Mangoldt in 1895. His strategy was to evaluate the integral

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) x^s \frac{ds}{s}$$

(where $a > 1$) in two different ways. The first is to substitute the expression for $-\zeta'(s)/\zeta(s)$ given by (2) and obtain $\psi(x)$. The second is to use the definition of $\xi(s)$ and its product expansion to write

$$\xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) = \Pi\left(\frac{s}{2}\right) \pi^{-s/2}(s-1)\zeta(s), \tag{6}$$

take the logarithmic derivative, and substitute the resulting expression for $-\zeta'(s)/\zeta(s)$ (given in the next proposition) in the integral to obtain an analytic formula. However, von Mangoldt's derivation of the asymptotic behavior of $\psi(x)$ for large x depends on first deriving the asymptotic behavior of the antiderivative

$$\int_0^x \psi(t)dt.$$

To derive a formula for the antiderivative we will evaluate the integral

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) x^s \frac{ds}{s+1}$$

in a similar way. In order to combine the two very similar derivations, we consider the integral

$$I(b) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)x^s\frac{ds}{s+b}, \tag{7}$$

where b is a non-negative real number.

2.9 The formula for $-\zeta'(s)/\zeta(s)$

Taking the logarithmic derivative of (6) and using the product representation (3) for $\Pi(s)$, we derive the formula for $\zeta'(s)/\zeta(s)$ given in the next proposition. Beside routine calculations, the proof only requires checking that the infinite series involved can be differentiated termwise.

Proposition 4. *Let b be a non-negative real number. Then*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{s+b}{(2n-b)(2n+s)} + \frac{s+b}{(s-1)(1+b)} + \sum_{\rho}\frac{s+b}{(\rho+b)(\rho-s)} - \frac{\zeta'(-b)}{\zeta(-b)}. \tag{8}$$

2.10 The formulas for $\psi(x)$ and $\int_0^x \psi(t)dt$

The next theorem gives analytic formulas for $\psi(x)$ and $\int_0^x \psi(t)dt$. The proof of this theorem also occupies a significant part of a chapter of Edwards' book. But it should be noted that if we ignore the fact that taking a limit inside an integral or an infinite sum must be justified, the proof amounts to routine calculations obtained by substituting (2) and (8) in (7). In this respect, a substantial part of the classical proof of the PNT could be described as the task of proving that some limits can be switched. Probably the most difficult part of the proof of this theorem is proving that when substituting (8) into (7), the term of (8) involving the roots ρ of $\xi(s)$ results in the limit

$$\lim_{h\to\infty}\sum_{|\mathrm{Im}(\rho)|\le h}\frac{x^\rho}{\rho+b}. \tag{9}$$

This was done by von Mangoldt, and he was only able to prove that the infinite series converges when the terms are arranged in increasing order of $|\mathrm{Im}(\rho)|$.

Theorem 4.

$$\psi(x) = I(0) = x - \frac{\zeta'(0)}{\zeta(0)} + \sum_{n=1}^{\infty}\frac{x^{-2n}}{2n} - \lim_{h\to\infty}\sum_{|\mathrm{Im}(\rho)|\le h}\frac{x^\rho}{\rho}$$

$$\int_0^x \psi(t)dt = xI(0) - xI(1)$$

$$= \frac{x^2}{2} - \frac{x\zeta'(0)}{\zeta(0)} + \frac{\zeta'(-1)}{\zeta(-1)} - \sum_{n=1}^{\infty}\frac{x^{1-2n}}{2n(2n-1)} - \lim_{h\to\infty}\sum_{|\mathrm{Im}(\rho)|\le h}\frac{x^{\rho+1}}{\rho(\rho+1)} \tag{10}$$

2.11 Re(ρ) < 1

The last significant step in the derivation of the asymptotic behavior of $\psi(x)$ (and so the proof of the PNT) is the proof that $\zeta(s)$ has no zeros on the line $\text{Re}(s) = 1$. This was proved by both Hadamard and de la Valleé Poussin independently in 1896.

Theorem 5.

$$\textit{If } \rho \textit{ is a zero of } \zeta(s)\textit{, then } \textit{Re}(\rho) < 1 \tag{11}$$

2.12 The asymptotic estimate for $\int_0^x \psi(t)dt$

From (10) and (11), it is easy to derive the asymptotic estimate for the antiderivative of $\psi(x)$.

Proposition 5.

$$\int_0^x \psi(t)dt \sim \frac{x^2}{2} \quad \textit{as } x \to \infty. \tag{12}$$

2.13 The asymptotic estimate for $\psi(x)$

The asymptotic behavior of $\psi(x)$ (and hence the Prime Number Theorem) now follows from the following general result on the antiderivative of a positive, increasing function.

Proposition 6. *Suppose that f is positive and increasing, and*

$$\int_0^x f(t)dt \sim \frac{x^2}{2} \quad \textit{as } x \to \infty.$$

Then

$$f(x) \sim x \quad \textit{as } x \to \infty.$$

3 Proofs of the theorems

3.1 Proof of Theorem 1

The starting point is the extension of the factorial (3). The integral representation

$$\Pi(s) = \int_0^\infty e^{-s} x^s dx. \tag{13}$$

valid for $\text{Re}(s) > -1$, was also known to Euler. Using (13) with s replaced by $s-1$, making the substitution $x \mapsto nx$, and summing over $n \geq 1$, we obtain the integral representation

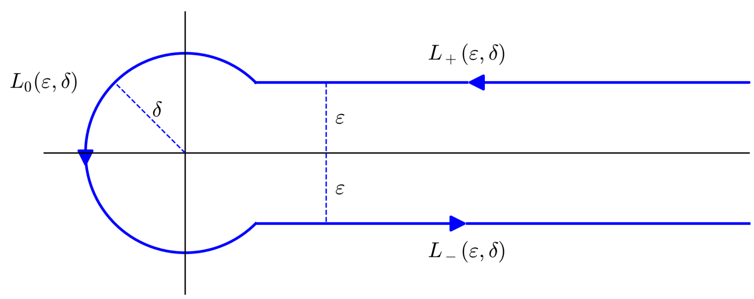
$$\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \Pi(s-1) \sum_{n=1}^\infty \frac{1}{n^s}, \tag{14}$$

valid for $s > 1$.

In view of the integral representation (14), Riemann considers the contour integral

$$\int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z},$$

where the integral is along the Hankel path $C = C(\delta, \varepsilon)$ that starts at distance ε above the positive real axis at infinity, circles counterclockwise around the origin with radius $\delta > \varepsilon$, and returns to infinity traveling at distance ε below the positive real axis (see figure below).



The Hankel path $C(\delta, \varepsilon)$

It is not hard to see that the integral does not depend on the values of ε and δ (as long as they are small enough so that no singularities are enclosed). This can be proved by truncating the path at $x = R$ for some large R, then joining it via vertical segments to a similar path $C(\delta_1, \varepsilon_1)$, where $\delta_1 < \delta$, $\varepsilon_1 < \varepsilon$, in order to form a closed contour without singularities inside, so that by Cauchy's theorem the integral will be zero, then let $R \to \infty$ and notice that the integral over the vertical segments will go to zero. This will prove that the integral over $C(\delta, \varepsilon)$ is the same as the integral over $C(\delta_1, \varepsilon_1)$.

On L_+ we have $-z = -x - i\varepsilon$ and $\log(-x - i\varepsilon) \to \log|z| - i\pi$ as $\varepsilon \to 0$, while on L_-, $-z = -x + i\varepsilon$, and $\log(-x + i\varepsilon) \to \log|z| + i\pi$ as $\varepsilon \to 0$. So the integral over L_+ approaches $e^{-i\pi s} \int_0^\infty \frac{x^s}{e^x - 1} \frac{dx}{x}$ and the integral over L_- approaches $e^{i\pi s} \int_0^\infty \frac{x^s}{e^x - 1} \frac{dx}{x}$ as $\varepsilon, \delta \to 0$. Also, if we write $z = \delta e^{i\theta}$ on C_0, the integrand is bounded by $\frac{\delta^s}{|e^{\delta e^{i\theta}} - 1|} = \frac{\delta^{s-1}}{|e^{i\theta} + O(\delta)|}$ and so the integral over C_0 will approach zero as long as $\mathrm{Re}(s) > 1$. So we conclude that, for $\mathrm{Re}(s) > 1$,

$$\int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z} = 2i \sin(\pi s) \int_0^\infty \frac{x^s}{e^x - 1} \frac{dx}{x} = 2i \sin(\pi s) \zeta(s) \Pi(s-1).$$

But the left side of this equation is an entire function of z. Using standard identities for the factorial function, this leads to the definition of the Riemann zeta function as in the statement of the theorem:

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i} \int_C \frac{(-z)^s}{e^z - 1} \frac{dz}{z}.$$

The integral on the right side defines an entire function. So the only singularities can come from the simple poles of $\Pi(-s)$ at the positive integers. But for $s = 2, 3, 4, \ldots$ the function coincides with $\sum_{n\geq 1} 1/n^s$, so the simple poles of $\Pi(-s)$ must be cancelled by zeros of the integral, while at $s = 1$ we know that $\sum_{n\geq 1} 1/n^s$ diverges, so there must be a simple pole (with residue 1) coming from $\Pi(-s)$. So formula (4) defines a function for all complex values of s except $s = 1$, where it has a simple pole, and it coincides with the previous definition for $\mathrm{Re}(s) > 1$.

The Maclaurin expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

introduces the Bernoulli numbers B_n in the evaluation of $\zeta(s)$. For $s = -n$, $n = 0, 1, 2, 3, \ldots$, the integrals over L_+ and L_- will cancel each other, and the integral over L_0 as $\delta, \varepsilon \to 0$ can easily be evaluated to obtain

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}.$$

In particular, since $B_n = 0$ for n odd and greater than 1, we find $\zeta(-2n) = 0$ for all positive integers n.

3.2 Proof of Theorem 2

(a) Formula (14) in the proof of Theorem 1 that led to the analytic continuation of $\zeta(s)$ from $s > 1$ to the whole complex plane (except $s = 1$) was obtained by substituting nx for x in the integral representation (13) for $\Pi(s-1)$, and then summing over n. If we instead make the substitution $x \mapsto n^2\pi x$ in the integral representation for $\Pi(s/2 - 1)$ and sum, we find

$$\Pi\left(\frac{s}{2} - 1\right) \pi^{-s/2} \zeta(s) = \int_0^{\infty} \phi(x) x^{s/2} \frac{dx}{x}, \tag{15}$$

where

$$\phi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}.$$

The function $\phi(x)$ is related to the Jocobi theta function

$$G(u) = \vartheta(0; iu^2) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 u^2}$$

by

$$2\phi(u^2) + 1 = G(u)$$

and so the functional equation for the theta function [D1] $uG(u) = G(1/u)$ for $G(u)$ gives

$$\phi(x) = \frac{1}{\sqrt{x}}\phi\left(\frac{1}{x}\right) + \frac{1}{\sqrt{x}} - \frac{1}{2}. \tag{16}$$

Differentiation of this identity gives

$$\phi(1) + 4\phi'(1) + \frac{1}{2} = 0. \tag{17}$$

By splitting the integral (15) from 0 to 1 and 1 to ∞, and then using the functional equation (16), some routine calculations give

$$\begin{aligned} & \Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2}\zeta(s) \\ = & \int_1^\infty \phi(x)\left(x^{s/2} + x^{(1-s)/2}\right)\frac{dx}{x} - \left(\frac{1}{s} + \frac{1}{1-s}\right). \end{aligned} \tag{18}$$

The integral on the right side of (18) is an entire function of s. So multiplying the equation by $s(s-1)/2$ we get an entire function, denoted by $\xi(s)$:

$$\begin{aligned} \xi(s) &= (s-1)\Pi\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) \\ &= \frac{1}{2} - \frac{s(1-s)}{2}\int_1^\infty \phi(x)\left(x^{s/2} + x^{(1-s)/2}\right)\frac{dx}{x}. \end{aligned} \tag{19}$$

The right side of (19) is invariant under $s \to 1-s$, and so we obtain the functional equation

$$\xi(s) = \xi(1-s).$$

(b) We show that $\zeta(s)$ (and hence $\xi(s)$) has no zeros for $\mathrm{Re}(s) > 1$. Let $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > 1$, and let $(p_n : n \geq 1)$ be the prime numbers. Let $a_n = (p_n^s - 1)^{-1}$, so that $1 + a_n = (1 - p_n^{-s})^{-1}$. Since

$$\sum_{n=1}^\infty |a_n| = \sum_{n=1}^\infty \frac{1}{|p_n^s - 1|} \leq \sum_{n=2}^\infty \frac{1}{n^\sigma - 1} < \infty,$$

we conclude from basic results on infinite products [Appendix C] that

$$\sum_{n=1}^\infty \log(1 + a_n) = \log \prod_{n=1}^\infty \left(1 - \frac{1}{p_n^s}\right)^{-1} = L \in \mathbb{C},$$

and so

$$\zeta(s) = \prod_{n=1}^\infty \left(1 - \frac{1}{p_n^s}\right)^{-1} = e^L \neq 0.$$

Using the functional equation $\xi(s) = \xi(1-s)$ it follows that $\xi(s)$ has no zeros outside $0 \leq \mathrm{Re}(s) \leq 1$.¡br/¿¡br/¿

(c) Using the estimates

$$\phi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x} \leq \sum_{n=1}^{\infty} e^{-\pi n x} = \frac{1}{e^{\pi x} - 1}$$

and

$$|\phi'(x)| = \pi \sum_{n=1}^{\infty} n^2 e^{-\pi n^2 x} \leq \pi \sum_{n=1}^{\infty} n^2 e^{-\pi n x} = \pi e^{-\pi x} \frac{1 + e^{-\pi x}}{(1 - e^{-\pi x})^3},$$

we see that for every s,

$$|x^s \phi(x)| \to 0 \text{ and } |x^s \phi'(x)| \to 0 \text{ as } x \to \infty.$$

Integrating (19) by parts twice and using (17) we then find

$$\begin{aligned} \xi(s) &= 2 \int_1^{\infty} \left(x^{s/2-1/2} + x^{-s/2} \right) \left(x^{3/2} \phi'(x) \right)' dx \\ &= 4 \int_1^{\infty} x^{-1/4} \cosh \left(\frac{1}{2} \left(s - \frac{1}{2} \right) \log x \right) \left(x^{3/2} \phi'(x) \right)' dx. \end{aligned}$$

Using the Maclaurin expansion for cosh, this gives

$$\xi(s) = \sum_{n=0}^{\infty} c_n \left(s - \frac{1}{2} \right)^{2n},$$

or

$$\xi \left(\frac{1}{2} + it \right) = \sum_{n=0}^{\infty} (-1)^n c_n t^{2n},$$

where

$$c_n = \frac{4}{(2n)!} \int_1^{\infty} x^{-1/4} \left(\frac{\log x}{2} \right)^{2n} \left(x^{3/2} \phi'(x) \right)' dx.$$

We find

$$\frac{d}{dx} \left(x^{3/2} \phi'(x) \right) = \pi x^{1/2} \sum_{n=1}^{\infty} n^2 \left(\pi x n^2 - \frac{3}{2} \right) e^{-\pi n^2 x}$$

and so all coefficients c_n are positive.

3.3 Proof of Theorem 3

It is reasonable to expect that in order to prove that a product

$$\prod_{\rho} \left(1 - \frac{s}{\rho} \right)$$

converges we need to have an estimate of how many zeros ρ there are. The first step in the proof will be to put a bound on the growth of $\xi(s)$, as described in the next result.

Lemma 1. *For all large enough R and $|s-1/2| \le R$, we have $|\xi(s)| \le R^R$.*

This will be used to provide the estimate on the number of zeros.

Lemma 2. *For $R > 0$, let $n(R)$ be the number of zeros of $\xi(s)$ such that $|s-1/2| \le R$. Then $n(R) \le 3R \log R$ for all large enough R.*

The above estimate of the number of zeros of $\xi(s)$ allows us to prove the next result.

Lemma 3. *Let $\rho_k : k = 1, 2, 3, \dots$ be the zeros of $\xi(s)$ ordered so that $|\rho_k - 1/2| \le |\rho_{k+1} - 1/2|$. Let $\varepsilon > 0$ be given. Then the series*

$$\sum_{k=1}^{\infty} \frac{1}{|\rho_k - 1/2|^{1+\varepsilon}}$$

converges, and so in particular the infinite product

$$P(s) = \prod_{k=1}^{\infty} \left(1 - \frac{(s-1/2)^2}{(\rho - 1/2)^2}\right)$$

converges.

The infinite product $P(s)$ has exactly the same zeros as $\xi(s)$. We will show that in fact it is the same as $\xi(s)$, up to a multiplicative constant. Our main tool is the next result from complex analysis, that can be thought of as an analogue of Liouville's theorem when an entire function is also even.

Lemma 4. *Suppose that $f(s)$ is entire and even and for each $\varepsilon > 0$, $Re(f(s)) \le \varepsilon |s|^2$ for all large enough $|s|$. Then $f(s)$ is a constant function.*

The next result provides the asymptotic estimate needed to make use of Lemma 4. Its proof depends on the estimate given by Lemma 1

Lemma 5. *Let $\varepsilon > 0$ be given. Then for all large enough $|s-1/2|$,*

$$Re \log \frac{\xi(s)}{P(s)} \le \left| s - \frac{1}{2} \right|^{1+\varepsilon}.$$

We can now derive the product expansion for $\xi(s)$ by applying Lemma 4 to the function $F(s) = \xi(s)/P(s)$.

3.4 Proof of Theorem 4

We will often need to consider the integral

$$F_h(x, \beta) = \frac{1}{2\pi i} \int_{a-ih}^{a+ih} \frac{x^{s-\beta}}{s-\beta} ds$$

where $a > 0$, $x > 0$, $h > 0$ and β is a complex number with $a > \mathrm{Re}(\beta)$. The next lemma describes the rate of convergence of $F_h(x, 0)$ as $h \to \infty$, and it will be used to justify taking limits inside integrals or infinite series. These estimates are derived by evaluating the integral along suitable large rectangles and making use of the residue theorem.

Lemma 6.

$$F_h(1,0) = \frac{1}{\pi}\tan^{-1}\left(\frac{h}{a}\right)$$

$$\textit{If } 0 < x < 1, \textit{ then} \qquad |F_h(x,0)| \le \frac{x^a}{\pi h |\log x|}. \tag{20}$$

$$\textit{If } x > 1, \textit{ then} \qquad |F_h(x,0) - 1| \le \frac{x^a}{\pi h \log x}. \tag{21}$$

In particular,

$$\lim_{h\to\infty} \frac{1}{2\pi i}\int_{a-ih}^{a+ih} x^s \frac{ds}{s} = \begin{cases} 0 & \textit{if } 0 < x < 1 \\ \frac{1}{2} & \textit{if } x = 1 \\ 1 & \textit{if } x > 1 \end{cases} \quad , \qquad a > 0. \tag{22}$$

The next two lemmas are simple consequences of the above estimates. They will be needed to justify taking some limits inside integrals.

Lemma 7.

$$\textit{If } x > 1 \textit{ and } a > Re(\beta), \textit{ then} \qquad \lim_{h\to\infty} F_h(x,\beta) = 1.$$

Lemma 8. *Suppose* $x > 1, d > c \ge 0$. *Then*

$$\left|\frac{1}{2\pi i}\int_{a+ic}^{a+id} \frac{x^s}{s} ds\right| \le K \frac{x^a}{(a+c)\log x}, \textit{ where } K = \sqrt{2}\left(\frac{1}{\pi} + \frac{1}{4}\right).$$

The next lemma is obtained by carrying out the first substitution of $-\zeta'(s)/\zeta(s)$ in $I(b)$, as mentioned in section 2.8, using equation (2).

Lemma 9.

$$I(0) = \psi(x) = \sum_{n<x} \Lambda(n) \tag{23}$$

$$I(1) = \frac{1}{x}\sum_{n<x} n\Lambda(n) \tag{24}$$

The analytic formulas for $\psi(x)$ and its antiderivative will now be derived by substituting formula (8) into $I(b)$. There are four terms in (8). The calculations, and the switching of infinite sums and integral, will be relatively straightforward for all terms except the one involving the sum of the roots ρ. That term will produce the limit

$$\lim_{h\to\infty} \sum_\rho \frac{x^\rho}{\rho + b} F_h(x,\rho),$$

because the infinite series (that is locally uniformly convergent) can be integrated termwise on finite paths. So we know that the limit exists, since all other terms in the formula obtained by substituting (8) into $I(b)$ exist. But the difficult part (as mentioned earlier) will be in proving that the same limit is equal to

$$\lim_{h\to\infty} \sum_{|\mathrm{Im}(\rho)\leq h} \frac{x^\rho}{\rho+b}.$$

One ingredient of the proof is the estimate of the vertical density of the roots of $\xi(s)$ given in the next lemma. Note that this formula can be considered a refinement of the formula for $n(R)$ given in Lemma 2, because it is easy to derive that formula if we have the vertical density estimate. But the proof of the vertical density given in the next lemma depends on the product formula for $\xi(s)$, and the formula for $n(R)$ was needed in order to prove the product formula. We will also need Stirling's formula for $\Pi(s)$, and in fact it will be necessary to use not just the dominant term, but also the fact that the relative error is of order $1/|s|$ for large $|s|$.

Lemma 10. *Let $D(T)$ be the number of roots of $\xi(s)$ with imaginary part between T and $T+1$. Then*

$$D(T) \leq 2\log T \qquad \textit{for all large } T. \tag{25}$$

The last step in the proof of Theorem 4 is the derivation of the term that results from the sum involving the roots of $\xi(s)$ when substituting (8) into $I(b)$. The main idea in von Mangoldt's proof is to split the difference

$$\sum_\rho \frac{x^\rho}{\rho+b} F_h(x,\rho) - \sum_{|\mathrm{Im}(\rho)|\leq h} \frac{x^\rho}{\rho+b}$$

as

$$\left(\sum_\rho \frac{x^\rho}{\rho+b} F_h(x,\rho) - \sum_{|\mathrm{Im}(\rho)|\leq h} \frac{x^\rho}{\rho+b} F_h(x,\rho)\right)$$
$$+ \quad \left(\sum_{|\mathrm{Im}(\rho)|\leq h} \frac{x^\rho}{\rho+b} F_h(x,\rho) - \sum_{|\mathrm{Im}(\rho)|\leq h} \frac{x^\rho}{\rho+b}\right)$$

and then use the estimate for $D(T)$ to prove that the differences in both parentheses go to zero as $h\to\infty$. The term that is added and subtracted is what Edwards calls a "diagonal" term.

Lemma 11.

$$\lim_{h\to\infty} \sum_\rho \frac{x^\rho}{\rho+b} F_h(x,\rho) = \lim_{h\to\infty} \sum_{|\mathrm{Im}(\rho)|\leq h} \frac{x^\rho}{\rho+b}$$

We are now ready to substitute (8) into (7). Since the series in (8) converge locally uniformly, we can integrate them termwise over finite paths and we find

$$\begin{aligned}
& \frac{1}{2\pi i}\lim_{h\to\infty}\int_{a-ih}^{a+ih}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)x^s\frac{ds}{s+b} \\
= & \frac{1}{2\pi i}\lim_{h\to\infty}\int_{a-ih}^{a+ih}\left(\frac{s+b}{(s-1)(1+b)}-\frac{\zeta'(-b)}{\zeta(-b)}+\sum_{n=1}^{\infty}\frac{s+b}{(2n-b)(2n+s)}\right. \\
+ & \left.\sum_{\rho}\frac{s+b}{(\rho+b)(\rho-s)}\right)x^s\frac{ds}{s+b} \\
= & \frac{1}{2\pi i(1+b)}\int_{a-i\infty}^{a+i\infty}\frac{x^s}{s-1}ds+\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(-\frac{\zeta'(-b)}{\zeta(-b)}\right)x^s\frac{ds}{s+b} \\
+ & \lim_{h\to\infty}\sum_{n=1}^{\infty}\frac{1}{(2n-b)}\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\frac{x^s}{s+2n}ds+\lim_{h\to\infty}\sum_{\rho}\frac{1}{(\rho+b)}\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\frac{x^s}{\rho-s}ds \\
= & \frac{x}{1+b}-x^{-b}\frac{\zeta'(-b)}{\zeta(-b)}+\lim_{h\to\infty}\sum_{n=1}^{\infty}\frac{1}{(2n-b)}\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\frac{x^s}{s+2n}ds \\
+ & \lim_{h\to\infty}\sum_{\rho}\frac{1}{(\rho+b)}\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\frac{x^s}{\rho-s}ds
\end{aligned}$$

where we have used Lemma 7 to evaluate the first two terms. We now need to show that we can switch the limit with the two infinite series. First consider the series in n. We will show that the it converges uniformly in h. We find

$$\begin{aligned}
& \int_{a-ih}^{a+ih}\frac{x^{s+2n}}{s+2n}ds=\int_0^h\left(\frac{x^{a+2n+iy}}{a+2n+iy}+\frac{x^{a+2n-iy}}{a+2n-iy}\right)dy \\
= & 2\mathrm{Re}\int_0^h\frac{x^{a+2n+iy}}{a+2n+iy}dy=2\mathrm{Re}\int_{a+2n}^{a+2n+ih}\frac{x^t}{t}dt
\end{aligned}$$

and so using Lemma 8,

$$\left|\int_{a-ih}^{a+ih}\frac{x^{s+2n}}{s+2n}ds\right|\le 2\left|\int_{a+2n}^{a+2n+ih}\frac{x^t}{t}dt\right|\le\frac{Kx^{a+2n}}{(a+2n)\log x}$$

for a constant K. It follows that the n-th term of the series in n is bounded by c/n^2 for a constant c independent of h. So the series converges uniformly in h, and it follows from Lemma 7 that

$$\lim_{h\to\infty}\sum_{n=1}^{\infty}\frac{1}{2n-b}\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\frac{x^s}{s+2n}ds=\sum_{n=1}^{\infty}\frac{1}{2n-b}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{x^s}{s+2n}ds=\sum_{n=1}^{\infty}\frac{x^{-2n}}{2n-b}.$$

So we have now shown that

$$I(b)=\frac{x}{1+b}-x^{-b}\frac{\zeta'(-b)}{\zeta(-b)}+\sum_{n=1}^{\infty}\frac{x^{-2n}}{2n-b}-\lim_{h\to\infty}\sum_{\rho}\frac{x^{\rho}}{\rho+b}F_h(x,\rho) \qquad (26)$$

and using Lemma 11 we obtain the formula for $I(b)$

$$I(b) = \frac{x}{1+b} - x^{-b}\frac{\zeta'(-b)}{\zeta(-b)} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n-b} - \lim_{h\to\infty} \sum_{|\mathrm{Im}(\rho)|\leq h} \frac{x^\rho}{\rho+b}.$$

Setting $b = 0$ we find, according to (23), van Mangoldt's formula for $\psi(x)$

$$\begin{aligned} I(0) &= \psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} - \lim_{h\to\infty} \sum_{|\mathrm{Im}(\rho)\leq h} \frac{x^\rho}{\rho} \\ &= x - \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2}\log\left(\frac{x^2}{x^2-1}\right) - \lim_{h\to\infty} \sum_{|\mathrm{Im}(\rho)\leq h} \frac{x^\rho}{\rho} \end{aligned}$$

while setting $b = 1$ we find, according to (24),

$$\begin{aligned} I(1) &= \frac{1}{2x}\left(\sum_{n<x} n\Lambda(n) + \sum_{n\leq x} n\Lambda(n)\right) \\ &= \frac{x}{2} - \frac{\zeta'(-1)}{x\zeta(-1)} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n-1} - \lim_{h\to\infty} \sum_{|\mathrm{Im}(\rho)\leq h} \frac{x^\rho}{\rho+1} \end{aligned}$$

and the formula for the antiderivative of $\psi(x)$ follows by computing $xI(0) - xI(1)$.

3.5 Proof of Theorem 5

Let $F(s) = \sum_p \frac{1}{p^s}$, for $\mathrm{Re}(s) > 1$. We claim that $F(\sigma) + \log(\sigma - 1)$ is bounded as $\sigma \to 1^+$. Using the representation

$$\Pi(s) = \lim_{n\to\infty} \frac{n!(n+1)^s}{(s+1)\cdots(s+n)}$$

we find

$$\lim_{s\to 1}(s-1)\Pi(-s) = -1.$$

Since

$$\frac{1}{2\pi i}\int_C \frac{-z}{e^z-1}\frac{dz}{z} = -1$$

(where C is the Hankel path from Chapter 1), we find from the integral representation

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i}\int_C \frac{(-z)^s}{e^z-1}\frac{dx}{z}$$

that

$$\lim_{s\to 1}(s-1)\zeta(s) = 1.$$

As a consequence of Euler's product formula [A2], we find

$$\log \zeta(s) = F(s) + B(s), |B(s)| \leq \frac{1}{2} \text{ for } \mathrm{Re}(s) > 1.$$

So

$$\log((\sigma - 1)\zeta(\sigma)) = \log(\sigma - 1) + F(\sigma) + B(\sigma) \to 0 \text{ as } \sigma \to 1^+$$

and the claim follows.

Suppose now $t \neq 0$ is such that $\zeta(1 + it) = 0$. Let $\delta > 0$ be such that $1 - \cos\delta > 0$ and $\cos(2\delta) > 0$. Consider the set

$$A = \{p \text{ prime} : |(2n + 1)\pi - t \log p| < \delta \text{ for some } n \in \mathbb{Z}\}$$

Let

$$F_1(\sigma) = \sum_{p \in A} \frac{1}{p^\sigma}, \quad F_2(\sigma) = \sum_{p \notin A} \frac{1}{p^\sigma}, \quad \sigma > 1.$$

Since $\zeta(s)/(s-1-it)$ is analytic, using $s = \sigma + it$ we find that $\zeta(\sigma + it)/(\sigma - 1)$ is bounded as $\sigma \to 1^+$, and so $\mathrm{Re} \log \zeta(\sigma + it) - \log(\sigma - 1)$ is bounded. This implies

$$\mathrm{Re}F(\sigma + it) - \log(\sigma - 1) = \sum_p \frac{\cos(t \log p)}{p^\sigma} - \log(\sigma - 1) \text{ is bounded as } \sigma \to 1^+.$$

If $p \notin A$, then $\cos(t \log p) \geq 1 - \cos(\delta)$, and so

$$\sum_p \frac{\cos(t \log p)}{p^\sigma} \geq -\sum_{p \in A} \frac{1}{p^\sigma} - \cos(\delta) \sum_{p \notin A} \frac{1}{p^\sigma} = -F_1(\sigma) - \cos(\delta) F_2(\sigma).$$

Hence

$$-F_1(\sigma) - \cos(\delta) F_2(\sigma) - \log(\sigma - 1) \leq \sum_p \frac{\cos(t \log p)}{p^\sigma} - \log(\sigma - 1) \text{ is bounded as } \sigma \to 1^+. \tag{27}$$

According to the claim,

$$F(\sigma) + \log(\sigma - 1) = F_1(\sigma) + F_2(\sigma) + \log(\sigma - 1) \text{ is bounded as } \sigma \to 1^+. \tag{28}$$

Adding (27) and (28), we find

$$(1 - \cos(\delta)) F_2(\sigma) \text{ is bounded as } \sigma \to 1^+.$$

So we conclude that $F_2(\sigma)$ is bounded as $\sigma \to 1^+$. But $\zeta(s)$ is analytic at $s = 1 + 2it$, and so $\mathrm{Re}F(\sigma + 2it)$ is bounded as $\sigma \to 1^+$. If $p \in A$, then $|2(2n + 1)\pi - 2t \log p| < 2\delta$ for some n, and so $\cos(2t \log p) \geq \cos(2\delta)$. So

$$\mathrm{Re}F(\sigma + 2it) \geq \cos(2\delta) F_1(\sigma) - F_2(\sigma)$$

implies that $F_1(\sigma)$ is also bounded as $\sigma \to 1^+$. But this contradicts the fact that $F(\sigma) = F_1(\sigma) + F_2(\sigma)$ is unbounded as $\sigma \to 1^+$.

4 Proofs of the propositions

4.1 Proof of Proposition 1

Let $d\theta$ be the point measure that assigns weight $\log p$ to prime p. Then

$$\int_0^x \frac{d\theta(t)}{\log t} = \sum_{p<x} \frac{\log p}{\log p} = \pi(x).$$

So for $x < y$,

$$\pi(y) - \pi(x) = \int_x^y \frac{d\theta(t)}{\log t} = \frac{\theta(y)}{\log y} - \frac{\theta(x)}{\log x} + \int_x^y \theta(t)\left(-\frac{1}{\log t}\right)' dt. \tag{29}$$

Suppose that $\theta(x) \sim x$ for large x. Let $\epsilon > 0$ be given, and take x large enough so that

$$(1-\epsilon)x < \theta(x) < (1+\epsilon)x.$$

Then from (29) we find

$$\begin{aligned}
\pi(y) - \pi(x) \quad &\leq \quad 2\epsilon\frac{x}{\log x} + (1+\epsilon)\frac{y}{\log y} - (1+\epsilon)\frac{x}{\log x} + (1+\epsilon)\int_x^y t\left(-\frac{1}{\log t}\right)' dt \\
&= \quad 2\epsilon\frac{x}{\log x} + (1+\epsilon)\left.\frac{t}{\log t}\right|_x^y - (1+\epsilon)\left.\frac{t}{\log t}\right|_x^y + (1+\epsilon)\int_x^y \frac{dt}{\log t} \\
&= \quad 2\epsilon\frac{x}{\log x} + (1+\epsilon)(\mathrm{Li}(y) - \mathrm{Li}(x)).
\end{aligned}$$

So we find

$$\frac{\pi(y)}{\mathrm{Li}(y)} \leq \frac{\pi(x)}{\mathrm{Li}(y)} + \frac{2\epsilon x}{\mathrm{Li}(y)\log x} - (1+\epsilon)\frac{\mathrm{Li}(x)}{\mathrm{Li}(y)} + 1 + \epsilon.$$

For a fixed x, when y is large enough, we get

$$\frac{\pi(y)}{\mathrm{Li}(y)} \leq 1 + 2\epsilon.$$

In a similar way,

$$\frac{\pi(y)}{\mathrm{Li}(y)} \geq 1 - 2\epsilon.$$

So we conclude that

$$\lim_{x\to\infty} \frac{\pi(x)}{\mathrm{Li}(x)} = 1.$$

4.2 Proof of proposition 2

By definition,

$$\int_0^\infty \frac{d\psi(x)}{x^s} = \sum_{n=2}^\infty \frac{1}{n^s}\Lambda(n).$$

The zeta function

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1$$

converges locally uniformly for $\mathrm{Re}(s) > 1$, and using Euler's product formula we obtain by logarithmic differentiation

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1} = \sum_p \log p \sum_{n=1}^\infty p^{-sn} = \sum_{n=2} \frac{1}{n^s}\Lambda(n) = \int_0^\infty \frac{1}{x^s} d\psi, \quad \mathrm{Re}(s) > 1.$$

4.3 Proof of Proposition 3

Suppose that $\psi(x) \sim x$ as $x \to \infty$. Note that

$$\psi(x) = \sum_{n=1}^\infty \theta\left(x^{1/n}\right),$$

and the sum is finite, because $\theta\left(x^{1/n}\right)$ is zero as soon as $2^n > x$. Since $\theta(x) = 0$ for $n > \log x/\log 2$, and $\theta(x) \geq \theta\left(x^{1/n}\right)$ for $n \geq 2$, we find

$$\psi(x) \leq \theta(x) + \frac{\log x}{\log 2}\theta\left(x^{1/2}\right).$$

So

$$\psi(x) - \frac{\theta\left(x^{1/2}\right)\log x}{\log 2} < \theta(x) < \psi(x). \tag{30}$$

Note that since $\psi(x) \sim x$, if $\epsilon > 0$,

$$\frac{\theta(x)}{x^{1+\epsilon}} \leq \frac{\psi(x)}{x^{1+\epsilon}} \to 0 \text{ as } x \to \infty.$$

So for small $\epsilon > 0$,

$$\frac{\theta\left(x^{1/2}\right)\log x}{x} = \frac{\theta\left(x^{1/2}\right)}{\left(x^{1/2}\right)^{1+\epsilon}} \frac{\log x}{\left(x^{1/2}\right)^{1-\epsilon}} \to 0 \text{ as } x \to \infty.$$

Dividing (30) by x, we conclude that

$$\lim_{x\to\infty} \frac{\theta(x)}{x} = 1.$$

4.4 Proof of Proposition 4

Taking the logarithmic derivative of (6), and using the product formula (3) for $\Pi(s)$, we find

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\left[\frac{1}{2}\log\left(1+\frac{1}{n}\right) - \frac{1}{2n+s}\right] - \frac{1}{2}\log\pi + \frac{1}{s-1} + \sum_{\rho}\frac{1}{\rho - s}. \quad (31)$$

Subtracting from (31) the same equation evaluated at $s = -b$, we find (8).

To justify the pointwise differentiation of infinite series, it is enough to show that the two infinite series in (31) converge locally uniformly. We find

$$\frac{1}{2}\log\left(1+\frac{1}{n}\right) - \frac{1}{2n+s} = \frac{1}{2n^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{n^k}\left(\frac{s^{k+1}}{2^{k+1}} - \frac{1}{k+2}\right)$$

and so

$$\left|\frac{1}{2}\log\left(1+\frac{1}{n}\right) - \frac{1}{2n+s}\right| \leq \frac{c}{n^2}$$

for some constant c, all s in a compact set, and all large enough n. This proves that the pointwise differentiation of the first series is justified.

For the second series, we need to pair terms with ρ and $(1-\rho)$. Then we find

$$\frac{1}{s-\rho} + \frac{1}{s-(1-\rho)} = \frac{2(s-1/2)}{(s-1/2)^2 - (\rho - 1/2)^2}$$

and so

$$\left|\frac{1}{s-\rho} + \frac{1}{s-(1-\rho)}\right| \leq \frac{K}{|\rho - 1/2|^2}$$

for a constant K, s in a compact set, and $|\rho - 1/2|$ large enough. Then the series converges by Lemma 3 .

4.5 Proof of Proposition 5

From (10), we find

$$\frac{1}{x^2}\left(\int_0^x \psi(t)dt - \frac{x^2}{2}\right) = -\frac{\zeta'(0)}{x\zeta(0)} + \frac{\zeta'(-1)}{x^2\zeta(-1)} - \sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n-1)} - \lim_{h\to\infty}\sum_{|\mathrm{Im}(\rho)|\leq h}\frac{x^{\rho-1}}{\rho(\rho+1)}$$

Since this last series in ρ now converges absolutely and uniformly in x, we can evaluate the limit as $x \to \infty$ termwise and since $\mathrm{Re}(\rho) - 1 < 0$, we find

$$\lim_{x\to\infty}\frac{1}{x^2}\left(\int_0^x \psi(t)dt - \frac{x^2}{2}\right) = 0.$$

4.6 Proof of Proposition 6

Write $F(x) = \int_0^x f(t)dt$. Let $\epsilon > 0$ be given. Then for large enough x,

$$(1-\epsilon)\frac{x^2}{2} \le F(x) \le (1+\epsilon)\frac{x^2}{2}.$$

If x is large enough and $y = \beta x$, where $\beta > 1$, we have

$$(1-\epsilon)\frac{y^2}{2} \le f(y)(y-x) + (1+\epsilon)\frac{x^2}{2} = (y-x)f(y) + (1-\epsilon)\frac{x^2}{2} + \epsilon x^2,$$

and then we find

$$\frac{1}{2}(1-\epsilon)(y^2 - x^2) \le (y-x)f(y) + \epsilon x^2$$

and using $y = \beta x$,

$$\frac{f(y)}{y} \ge \frac{1}{2}(1-\epsilon)\frac{\beta+1}{\beta} - \frac{\epsilon}{\beta(\beta-1)}.$$

Hence

$$\liminf_{x\to\infty} \frac{f(x)}{x} \ge \frac{1}{2}(1-\epsilon)\frac{\beta+1}{\beta} - \frac{\epsilon}{\beta(\beta-1)}.$$

Letting ϵ go to zero, we find

$$\liminf_{x\to\infty} \frac{f(x)}{x} \ge \frac{\beta+1}{2\beta},$$

and letting β approach 1,

$$\liminf_{x\to\infty} \frac{f(x)}{x} \ge 1.$$

In a similar way, we find

$$\frac{f(x)}{x} \le \frac{1}{2}(1-\epsilon)(\beta+1) + \epsilon\frac{\beta^2}{\beta-1}$$

and so

$$\limsup_{x\to\infty} \frac{f(x)}{x} \le \frac{1}{2}(1-\epsilon)(\beta+1) + \epsilon\frac{\beta^2}{\beta-1}.$$

Letting ϵ go to zero, we find

$$\limsup_{x\to\infty} \frac{f(x)}{x} \le \frac{\beta+1}{2},$$

and then letting β approach 1,

$$\limsup_{x\to\infty} \frac{f(x)}{x} \le 1 \le \liminf_{x\to\infty} \frac{f(x)}{x}.$$

So we conclude that

$$\lim_{x\to\infty} \frac{f(x)}{x} = 1.$$

5 Proofs of the lemmas

5.1 Proof of Lemma 1

According to Theorem 2 d., the coefficients c_n for the power series of ξ as an even function of $s-1/2$ are positive real numbers. Hence if $|s-1/2| \leq R$, we find

$$|\xi(s)| = \left|\sum_{n=0}^{\infty} c_n \left(s - \frac{1}{2}\right)^{2n}\right| \leq \sum_{n=0}^{\infty} c_n \left|s - \frac{1}{2}\right|^{2n} \leq \sum_{n=0}^{\infty} c_n R^{2n} = \xi(R+1/2).$$

For a fixed $R > 0$, let $N \in \mathbb{N}$ be such that

$$\frac{1}{4} + \frac{R}{2} \leq N < \frac{1}{4} + \frac{R}{2} + 1.$$

Recall that

$$\xi(s) = (s-1)\Pi\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s).$$

Since $\xi(t) = \sum_{n=0}^{\infty} c_n (t-1/2)^{2n}$ is increasing for t real and $t \geq 1/2$, we have

$$\begin{aligned} & \xi\left(\frac{1}{2}+R\right) \leq \xi(2N) = (2N-1)\Pi(N)\pi^{-N}\zeta(2N) \leq (2N)N!\zeta(2) \\ \leq \quad & 2\zeta(2)N^{N+1} \leq 2\zeta(2)\left(\frac{1}{4}+\frac{R}{2}+1\right)^{1/4+R/2+2} \leq R^R \end{aligned}$$

for R large enough.

5.2 Proof of Lemma 2

If $R > 0$ and f is a function defined and analytic for $|z| \leq R$ and with $f(0) \neq 0$, let

$$Z(f;R) = \{z \in \mathbb{C} : f(z) = 0 \text{ and } |z| < R\}$$
$$Z_0(f;R) = \{z \in \mathbb{C} : f(z) = 0 \text{ and } |z| \leq R\}.$$

Then $Z_0(f;R) \subset Z(f;2R)$. If $z \in Z(f;2R)$, then $2R/|z| > 1$, so $\log(|2R/z|) > 0$, and we find

$$\sum_{z \in Z(f;2R)} \log\left|\frac{2R}{z}\right| \geq \sum_{z \in Z_0(f;R)} \log\left|\frac{2R}{z}\right| \geq \log(2)|Z_0(f;R)|,$$

because $2R/|z| \geq 2$ for $z \in Z_0(f;R)$. If f has no zeros on the circle $|z| = 2R$, Jensen's formula [Appendix F] applies and we find

$$\log|f(0)| + \log(2)|Z_0(f;R)| \leq \log|f(0)| + \sum_{z \in Z(f;2R)} \log\left|\frac{2R}{z}\right|$$

$$= \log\left(|f(0)| \prod_{z \in Z(f;2R)} \frac{2R}{|z|}\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(2Re^{i\theta})\right| d\theta. \tag{32}$$

Now let $f(z) = \xi(z + 1/2)$. Then $n(R) = |Z_0(f;R)|$, and by Lemma 1, we find

$$\left|f\left(2Re^{i\theta}\right)\right| = \left|\xi\left(\frac{1}{2} + 2Re^{i\theta}\right)\right| \leq (2R)^{2R}.$$

If ξ has no zeros on the circle $|s - 1/2| = 2R$, we find from (32)

$$\log\left|\xi\left(\frac{1}{2}\right)\right| + \log(2)n(R) \leq 2R\log(2R) = 2R\log R + 2R\log(2),$$

or

$$n(R) \leq \frac{2}{\log 2} R\log R + 2R - \frac{\log|\xi(1/2)|}{\log 2}$$

Let R_0 be such that the right side of the above inequality is $\leq 3R\log R$ for $R \geq R_0$. So $n(R) \leq 3R\log R$ if $R \geq R_0$ and there are no zeros on the circle $|s-1/2| = 2R$. If $R \geq R_0$ and there are some zeros on the circle $|s-1/2| = 2R$, we can find some $\epsilon_R > 0$ such that f has no zeros on $|s - 1/2| = 2(R + \epsilon)$ for all $0 < \epsilon \leq \epsilon_R$, and then

$$n(R) \leq n(R+\epsilon) \leq 3(R+\epsilon)\log(R+\epsilon)$$

for all $0 < \epsilon \leq \epsilon_R$. Letting $\epsilon \to 0$, we find $n(R) \leq 3R\log R$. So the inequality holds for all $R \geq R_0$.

5.3 Proof of Lemma 3

Define R_n implicitly by $4R_n \log R_n = n$. Then if $n(R)$ is as defined in the previous lemma, we find $n(R_n) \leq 3R_n \log R_n = 3n/4 < n$. This means that the n-th zero is outside the circle of radius R_n centered at $1/2$, so $|\rho_n - 1/2| > R_n$, and

$$\frac{1}{|\rho_n - 1/2|^{1+\epsilon}} < \frac{1}{R_n^{1+\epsilon}} = \frac{(4\log R_n)^{1+\epsilon}}{n^{1+\epsilon}} = \frac{(4\log R_n)^{1+\epsilon}}{n^{1+\epsilon/2}} \frac{1}{n^{1+\epsilon/2}}.$$

Since $\log R_n < \log 4 + \log R_n + \log\log R_n = \log(4R_n \log R_n) = \log n$, we find

$$\frac{(4\log R_n)^{1+\epsilon}}{n^{1+\epsilon/2}} < \frac{(4\log n)^{1+\epsilon}}{n^{1+\epsilon/2}} < 1$$

for large n, and so

$$\frac{1}{|\rho_n - 1/2|^{1+\epsilon}} < \frac{1}{n^{1+\epsilon/2}}$$

holds for all large enough n, and the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|\rho_n - 1/2|^{1+\epsilon}}$ converges.

5.4 Proof of Lemma 4

The proof depends on the following result, that provides a bound on $|f(s)|$ on a disk $|s| \le r$ if we are given a bound for $\mathrm{Re} f(s)$ on a larger disk $|s| \le R$.

Lemma A *Suppose $f(s)$ is analytic on $|s| \le R$, $f(0) = 0$, and $r < R$. If $Ref(s) \le M$ for $|s| \le R$, then*

$$|f(s)| \le \frac{2Mr}{R-r} \text{ for } |s| \le r.$$

Proof of Lemma A. Write $f(s) = u(s) + iv(s)$. Since $u(s)$ is harmonic, its maximum on $|s| \le R$ is achieved at the boundary. Let

$$\phi(s) = \frac{f(s)}{s(2M - f(s))}$$

Note that

$$u - 2M \le u \le M \le 2M - u,$$

and so

$$|u(s)| \le 2M - u(s).$$

Then we find

$$|2M - f(s)|^2 = (2M - u(s))^2 + v(s)^2 \ge u(s)^2 + v(s)^2 = |f(s)|^2.$$

Hence

$$|\phi(s)| \le \frac{1}{|s|} = \frac{1}{R} \text{ on } |s| \le R.$$

Then

$$|f(s)| = \left|\frac{2Ms\phi(s)}{1 + s\phi(s)}\right| \le \frac{2Mr/R}{1 - r/R} = \frac{2Mr}{R-r} \text{ on } |s| \le r. \blacksquare$$

To prove Lemma 4, suppose that $f(s)$ is entire and even, and for every $\epsilon > 0$, $\mathrm{Re} f(s) \le \epsilon|s|^2$ for large enough $|s|$. Assume first $f(0) = 0$. Write

$$f(s) = \sum_{n=2} a_n s^n.$$

Then given $r > 0$ and using Lemma A with $R = 2r$, we find

$$|a_n| = \left|\frac{1}{2\pi i}\int_{|s|=r} \frac{f(re^{i\theta})}{s^{n+1}} ds\right| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(re^{i\theta})|}{r^n} d\theta \le \frac{1}{2\pi}\frac{\epsilon 4r^2}{r^n} = \frac{2\epsilon}{\pi r^{n-2}}.$$

Since $n \ge 2$ and ϵ is arbitrary, we must have $a_n = 0$.

If $f(0) \ne 0$, we can consider $g(s) = f(s) - f(0)$, that satisfies the same growth condition, and conclude that $f(s) = f(0)$.

5.5 Proof of Lemma 5

To prove Lemma 5 we will make use of three lemmas.

Lemma B *If* $|z| \le 1/2$*, then* $-\mathrm{Re}\log(1-z) \le 2|z|$.

Proof of Lemma B.

$$-\log(1-z) = \int_0^z \frac{dw}{1-w}.$$

So

$$\begin{aligned} -\mathrm{Re}\log(1-z) &= \mathrm{Re}\int_0^z \frac{dw}{1-w} \\ &\le \left|\int_0^z \frac{dw}{1-w}\right| \le |z| \max\left\{\frac{1}{|1-w|} : |w| \le \frac{1}{2}\right\} \le 2|z| \quad \blacksquare \end{aligned}$$

Lemma C *Suppose* $a_n \ge 0$*, and* $\sum_{n=1}^{\infty} a_n$ *converges. For* $t > 0$*, let* $A(t) = \{n \in \mathbb{N} : a_n \le t\}$*. Then*

$$\lim_{t \to 0^+} \sum_{n \in A(t)} a_n = 0.$$

Proof of Lemma C. Let $\epsilon > 0$. Find m such that $\sum_{n=m}^{\infty} a_n < \epsilon$. If $a_n = 0$ for $1 \le n \le m$, then $\sum_{n \in A(t)} a_n \le \sum_{n=m}^{\infty} a_n < \epsilon$. Otherwise, let $\delta = \min\{a_n : a_n > 0, 1 \le n \le m\}$. Suppose $t < \delta$. If $n \le m$, then either $a_n = 0$, or $a_n \ge \delta > t$, and then $n \notin A(t)$, so

$$\sum_{n \in A(t)} a_n \le \sum_{n=m}^{\infty} a_n < \epsilon \quad \blacksquare$$

Lemma D *Suppose* z_n *is such that for every* $\epsilon > 0$*,* $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{1+\epsilon}}$ *converges. For* $R > 0$ *and* $s \in \mathbb{C}$*, let*

$$v_R(s) = Re \log \frac{1}{\prod_{|z_n| \ge 2R} \left(1 - \frac{s^2}{z_n^2}\right)}.$$

Then given $\epsilon > 0$*, if* $|s| = R$ *and* R *is large enough,* $v_R(s) \le R^{1+\epsilon}$.

Proof of Lemma D. Let $\epsilon > 0$, and suppose $|s| = R$. Then

$$\begin{aligned} v_R(s) &= -\mathrm{Re} \sum_{|z_n| \geq 2R} \log\left(1 - \frac{s^2}{z_n^2}\right) \\ &\leq 2 \sum_{|z_n| \geq 2R} \frac{R^2}{|z_n|^2} \qquad \text{from Lemma B} \\ &= 2 \sum_{|z_n| \geq 2R} \left(\frac{R}{|z_n|}\right)^{1-\epsilon} \left(\frac{R}{|z_n|}\right)^{1+\epsilon} \\ &\leq 2 \left(\frac{1}{2}\right)^{1-\epsilon} \sum_{|z_n| \geq 2R} \left(\frac{R}{|z_n|}\right)^{1+\epsilon} \\ &= 2^{\epsilon} R^{1+\epsilon} \sum_{|z_n| \geq 2R} \left(\frac{1}{|z_n|}\right)^{1+\epsilon}. \end{aligned}$$

From Lemma C, the sum will go to zero as $R \to \infty$, so $v_R(s) \leq R^{1+\epsilon}$ when $|s| = R$ and R is large enough.■

To prove Lemma 5, consider the function

$$F(s) = \frac{\xi(s)}{P(s)}. \tag{33}$$

For each zero ρ of $\xi(s)$, the factor $1 - (s - 1/2)/(\rho - 1/2) = 2(s - \rho)/(1 - 2\rho)$ in the product $P(s)$ will cancel the zero of $\xi(s)$ at ρ, and so $F(s)$ is entire and without zeros. Fix $R > 0$, and write

$$A_R(s) = \prod_{|\rho - 1/2| \leq 2R} \left(1 - \left(\frac{s - 1/2}{\rho - 1/2}\right)^2\right),$$

$$B_R(s) = \prod_{|\rho - 1/2| > 2R} \left(1 - \left(\frac{s - 1/2}{\rho - 1/2}\right)^2\right),$$

so that $P(s) = A_R(s)B_R(s)$. Then on the disk $|s - 1/2| \leq R$, $\xi(s)/A_R(s)$ is analytic and has no zeros, because the zeros of $\xi(s)$ are cancelled by those of $A_R(s)$, and $1/B_R(s)$ is analytic, because all the zeros of B_R are in $|s - 1/2| > 2R$. So we can consider

$$u_R(s) = \mathrm{Re} \log \frac{\xi(s)}{A_R(s)} = \log \frac{|\xi(s)|}{\displaystyle\prod_{|\rho - 1/2| \leq 2R} \left|1 - \frac{(s - 1/2)^2}{(\rho - 1/2)^2}\right|}.$$

$$v_R(s) = \mathrm{Re} \log \frac{1}{B_R(s)} = \log \frac{1}{\displaystyle\prod_{|\rho - 1/2| > 2R} \left|1 - \frac{(s - 1/2)^2}{(\rho - 1/2)^2}\right|},$$

both defined and harmonic on $|s-1/2| \leq R$, and

$$u_R(s) + v_R(s) = \operatorname{Re} \log F(s).$$

Now consider $u_R(s)$ on $|s-1/2| \leq 4R$. Then u_R is harmonic there except at the points $s = \rho$ with $2R < |\rho - 1/2| \leq 4R$. But at those points $u_R(s) \to -\infty$. If we draw closed disks of small radius ϵ around each point of the finite set of zeros of $\xi(s)$ in the disk $|s-1/2| \leq 4R$, then u_R is harmonic on the same disk with those small disks removed, and so it will achieve its maximum on the boundary. By taking the radius of the disks small enough, since $u \to -\infty$ on the boundary of each small disk, we conclude that the maximum of u is achieved on the circle $|s-1/2| = 4R$. But if $|s-1/2| = 4R$, then for $|\rho - 1/2| \leq 2R$, we have

$$\left|1 - \frac{s-1/2}{\rho-1/2}\right| \geq \left|\frac{s-1/2}{\rho-1/2}\right| - 1 = \frac{4R}{|\rho-1/2|} - 1 \geq \frac{4R}{2R} - 1 = 1.$$

So each factor in the denominator has absolute value at least 1, and

$$\begin{aligned} & \max\{u_R(s) : |s-1/2| = R\} \\ \leq\ & \max\{u_R(s) : |s-1/2| \leq 4R\} = \max\{u_R(s) : |s-1/2| = 4R \\ \leq\ & \max\{\log|\xi(s)| : |s-1/2| = 4R\} \\ \leq\ & \log(4R)^{4R} = 4R\log(4R) \leq R^{1+\epsilon} \end{aligned} \tag{34}$$

for large R.

We now prove the same inequality for $v_R(s)$. By taking $z_n = \rho - 1/2$ in Lemma D, we find that for all $\epsilon > 0$,

$$v_R(s) = \operatorname{Re} \log \frac{1}{B_R(s)} = \log \frac{1}{\prod_{|\rho-1/2|>2R} \left|1 - \frac{(s-1/2)^2}{(\rho-1/2)^2}\right|} \leq R^{1+\epsilon} \tag{35}$$

for $|s-1/2| = R$ and all large R. Let now $\epsilon > 0$ be given, and let $F(s) = \xi(s)/P(s)$ as given in (33). From (34) and (35), we find $\operatorname{Re}\log F(s) = u_R(s) + v_R(s) \leq 2|s-1/2|^{1+\epsilon/2} \leq |s-1/2|^{1+\epsilon}$ for large enough $|s-1/2|$. So we conclude that

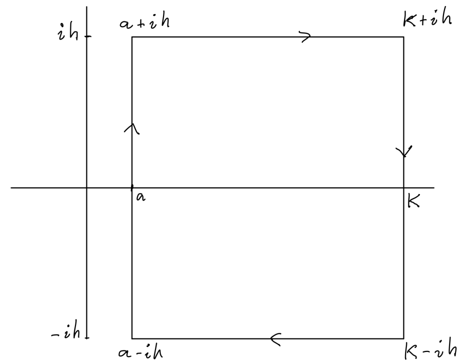
$$\operatorname{Re} \log F(s) \leq \left|s - \frac{1}{2}\right|^{1+\epsilon}$$

for all large enough $|s-1/2|$.

5.6 Proof of Lemma 6

$$F_h(1,0) = \frac{1}{2\pi i} \int_{a-ih}^{a+ih} \frac{ds}{s} = \frac{1}{2\pi} \int_{-h}^{h} \frac{a-it}{a^2+t^2} dt = \frac{a}{2\pi} \int_{-h}^{h} \frac{dt}{a^2+t^2} = \frac{1}{\pi} \tan^{-1}\left(\frac{h}{a}\right).$$

If $0 < x < 1$, consider the integral over the rectangle shown:



The integral over the closed path is zero, because the integrand is analytic inside. The integral over the top or bottom sides of the rectangle are easily seen to be bounded by

$$\frac{1}{2\pi}\frac{x^K - x^a}{h|\log x|},$$

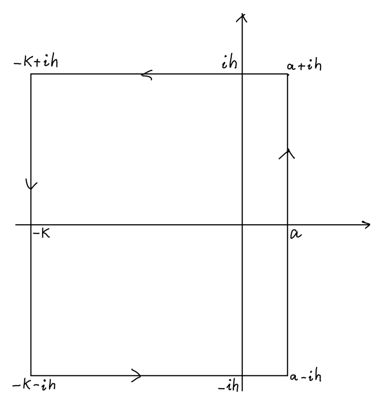
and the integral on the right side is bounded by

$$\frac{1}{2\pi}\frac{x^K}{K}(2h).$$

Letting $K \to \infty$, we find

$$|F_h(x,0)| = \left|\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\frac{x^s}{s}ds\right| \leq \frac{x^a}{\pi h|\log x|}.$$

If $x > 1$, consider the integral over the rectangle:



The integrand has a simple pole at $s = 0$, and the integral over the closed path is 1. The integral over the top or bottom is bounded by

$$\frac{1}{2\pi}\frac{x^a - x^{-K}}{h\log x}$$

and the integral over the left side is bounded by

$$\frac{1}{2\pi}\frac{2h}{Kx^K}.$$

Letting $K \to \infty$, we find the bound

$$|F_h(x,0)-1| = \left|\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\frac{x^s}{s}ds - 1\right| \le \frac{x^a}{\pi h \log x}.$$

5.7 Proof of Lemma 7

Consider the integral

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^s \frac{ds}{s-\beta},$$

where $x>1$, $\beta = \sigma + i\tau$ and $a > \sigma$. Let $t = s-\beta$. Then

$$\int_{a-ih}^{a+ih} x^s\frac{ds}{s-\beta} = x^\beta\int_{a-\sigma-i(h+\tau)}^{a-\sigma+i(h-\tau)} x^t\frac{dt}{t} = x^\beta\int_{a-\sigma-i(h+\tau)}^{a-\sigma+i(h+\tau)} x^t\frac{dt}{t} + x^\beta\int_{a-\sigma+i(h+\tau)}^{a-\sigma+i(h-\tau)} x^t\frac{dt}{t}.$$

The limit of the first term as $h\to\infty$ is $2\pi i x^\beta$, by (22). For the second term, we find

$$\begin{aligned}\left|x^\beta\int_{a-\sigma+i(h+\tau)}^{a-\sigma+i(h-\tau)} x^t\frac{dt}{t}\right| &\le x^{a-\sigma}\max\left\{\frac{1}{\sqrt{(a-\sigma)^2+y^2}} : h-\tau\le y\le h+\tau\right\}2\tau \\ &\le \frac{2\tau x^{a-\sigma}}{h-\tau} \to 0 \text{ as } h\to\infty.\end{aligned}$$

So we conclude that

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^s\frac{ds}{s-\beta} = x^\beta, \quad x>1, \quad a>\mathrm{Re}(\beta).$$

5.8 Proof of Lemma 8

Integrating by parts we find

$$\int_{a+ic}^{a+id}\frac{x^s}{s}ds = \frac{x^{a+id}}{(a+id)\log x} - \frac{x^{a+ic}}{(a+ic)\log x} + \frac{ix^a}{\log x}\int_c^d \frac{x^{it}}{(a+it)^2}dt. \qquad (36)$$

Using the inequality

$$\sqrt{a^2+c^2} \ge \frac{a+c}{\sqrt{2}},$$

we find

$$|a+id| \ge |a+ic| \ge \frac{a+c}{\sqrt{2}}$$

and so the first two terms on the right side of (36) are bounded by

$$\frac{x^a\sqrt{2}}{(a+c)\log x}.$$

The integral on the right side of (36) is bounded by

$$\left|\int_c^d \frac{x^{it}}{(a+it)^2}dt\right| \le \int_0^\infty \frac{du}{a^2+(c+u)^2} \le \int_0^\infty \frac{du}{a^2+c^2+u^2} = \frac{1}{\sqrt{a^2+c^2}}\frac{\pi}{2} \le \frac{\sqrt{2}}{a+c}\frac{\pi}{2}.$$

So we conclude that

$$\left|\frac{1}{2\pi i}\int_{a+ic}^{a+id}\frac{x^s}{s}ds\right| \le K\frac{x^a}{(a+c)\log x},$$

where

$$K = \sqrt{2}\left(\frac{1}{\pi}+\frac{1}{4}\right).$$

5.9 Proof of Lemma 9

Let $x > 2$ be fixed, and $m = \lceil x \rceil$, $a > 1$. Then for each $h > 0$,

$$\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)x^s\frac{ds}{s+b} = \frac{1}{2\pi i}\int_{a-ih}^{a+ih}\sum_{n=2}^{\infty}\Lambda(n)\frac{x^s}{n^s}\frac{ds}{s+b}$$
$$= \frac{1}{2\pi i}\int_{a-ih}^{a+ih}\sum_{n=2}^{m-1}\Lambda(n)\left(\frac{x}{n}\right)^s\frac{ds}{s+b} + \Lambda(m)\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\left(\frac{x}{m}\right)^s\frac{ds}{s+b}$$
$$+\frac{1}{2\pi i}\int_{a-ih}^{a+ih}\sum_{n=m+1}^{\infty}\Lambda(n)\left(\frac{x}{n}\right)^s\frac{ds}{s+b}.$$

The first term involves a finite sum and its limit as $h \to \infty$ is

$$\begin{aligned}\sum_{n=2}^{m-1}\Lambda(n)\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(\frac{x}{n}\right)^s\frac{ds}{s+b} &= \sum_{n=2}^{m-1}\Lambda(n)\frac{n^b}{2\pi i x^b}\int_{a+b-i\infty}^{a+b+i\infty}\left(\frac{x}{n}\right)^t\frac{dt}{t}\\ &= \frac{1}{x^b}\sum_{n=2}^{m-1}n^b\Lambda(n) = \frac{1}{x^b}\sum_{n<x}n^b\Lambda(n),\end{aligned}$$

where we have used (22). In a similar way, the limit of the second term is

$$\Lambda(m)\frac{1}{2}[x=m].$$

So if $b = 0$, the sum of the first two terms gives

$$\frac{1}{2}\sum_{n<x}\Lambda(n)+\frac{1}{2}\sum_{n<x}\Lambda(n)+ frac12\Lambda(n)[n=x] = \frac{1}{2}\left(\sum_{n<x}\Lambda(n)+\sum_{n\le x}\Lambda(n)\right) = \psi(x),$$

while if $b = 1$ we get

$$\frac{1}{2x}\left(\sum_{n<x} n\Lambda(n) + \sum_{n\le x} n\Lambda(n)\right)$$

For the third term, since the series converges uniformly on any halfplane $\mathrm{Re}(s) \ge a > 1$, we can integrate termwise on finite paths. Using $\Lambda(n) \le \log(n)$ and the estimate (20) we find

$$\begin{aligned} & \left|\int_{a-ih}^{a+ih} \sum_{n=m+1}^{\infty} \Lambda(n)\left(\frac{x}{n}\right)^s \frac{ds}{s+b}\right| \\ \le\ & \sum_{n=m+1}^{\infty} \Lambda(n)\frac{n^b}{x^b}\left|\int_{a+b-ih}^{a+b+ih} \left(\frac{x}{n}\right)^t \frac{dt}{t}\right| \\ \le\ & \sum_{n=m+1}^{\infty} \Lambda(n)\frac{x^a}{n^a \pi h \log(1+1/x)} \le \frac{1}{h}\sum_{n=m+1}^{\infty} \frac{c}{n^a} \end{aligned}$$

for some constant c. So the limit of the third term is zero as $h \to \infty$. This proves that

$$I(0) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) x^s \frac{ds}{s} = \psi(x) = \sum_{n<x} \Lambda(n)$$

and

$$I(1) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) x^s \frac{ds}{s+1} = \frac{1}{x}\left(\sum_{n<x} n\Lambda(n)\right).$$

5.10 Proof of Lemma 10

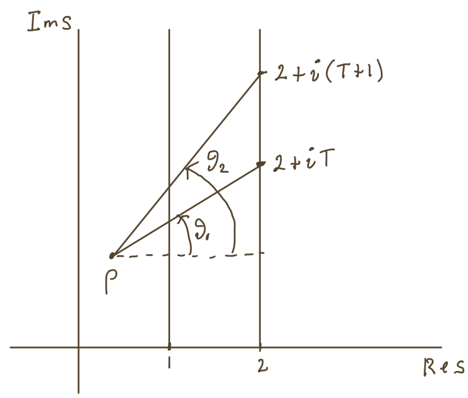
We first prove a lemma that will be needed in order to put an upper bound on the number of roots ρ in a horizontal strip of height 1.

Lemma E Let $\rho = \sigma + i\tau$ be a root of $\zeta(s)$ with $0 \le \sigma \le 1$, and let $T \ge 0$. Then

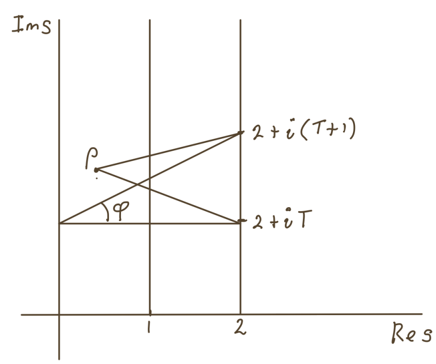
(a) $\mathrm{Im} \displaystyle\int_{2+iT}^{2+i(T+1)} \frac{ds}{s-\rho} \ge 0$

(b) If $T \le \tau \le T+1$, then $\mathrm{Im} \displaystyle\int_{2+iT}^{2+i(T+1)} \frac{ds}{s-\rho} \ge \tan^{-1}\left(\frac{1}{2}\right)$

Proof From a geometric point view, part (a) follows from the fact that the left side is the angle $\theta_2 - \theta_1$ in the following picture,



and part (b) from the fact that in case $T \le \tau \le T+1$, the same angle is at least the angle φ in the following picture.



(a) Let $s = \rho + re^{i\theta}$. Then (using Cauchy's theorem to replace the segment $[2+iT, 2+i(T+1)]$ with an arc centered at ρ), we find

$$\operatorname{Im} \int_{2+iT}^{2+i(T+1)} \frac{ds}{s-\rho} = \theta_2 - \theta_1,$$

where

$$\tan\theta_1 = \frac{T-\tau}{2-\sigma} < \frac{T+1-\tau}{2-\sigma} = \tan\theta_2.$$

So $\theta_1 < \theta_2$.

(b) Let $s = 2 + i(u + T)$, $\rho = \sigma + i(b + T)$. Then $0 \le \sigma \le 1$, $0 \le b \le 1$, and

$$\begin{aligned}
\operatorname{Im} \int_{2+iT}^{2+i(T+1)} \frac{ds}{s-\rho} &= \operatorname{Im} \int_0^1 \frac{idu}{2-\sigma+i(u-b)} \\
&= \operatorname{Im}\, i \int_0^1 \frac{2-\sigma-i(u-b)idu}{(2-\sigma)^2+(u-b)^2} \\
&= \int_0^1 \frac{2-\sigma}{(2-\sigma)^2+(u-b)^2} \\
&= \tan^{-1}\frac{b}{2-\sigma} - \tan^{-1}\frac{b-1}{2-\sigma} \\
&= \tan^{-1}\frac{2-\sigma}{(2-\sigma)^2+b(b-1)}
\end{aligned}$$

Now the lemma follows from the fact that the inequality $\sigma(2-\sigma)+b(1-b) \ge 0$ is equivalent to $\frac{2-\sigma}{(2-\sigma)^2+b(b-1)} \ge \frac{1}{2}$ ■

Since the Hadamard product

$$\xi(s) = \xi(0)\prod_\rho \left(1 - \frac{s}{\rho}\right)$$

was shown to converge locally uniformly, we can take the logarithmic derivative and then integrate termwise over finite paths. So using Lemma E, we find

$$\begin{aligned}
\operatorname{Im} \int_{2+iT}^{2+i(T+1)} \frac{\xi'(s)}{\xi(s)}ds = \operatorname{Im} \sum_\rho \int_{2+iT}^{2+i(T+1)} \frac{ds}{s-\rho} &\ge \sum_{T \le \operatorname{Im}(\rho) \le T+1} \int_{2+iT}^{2+i(T+1)} \frac{ds}{s-\rho} \\
&\ge D(T)\tan^{-1}\left(\frac{1}{2}\right). \qquad (37)
\end{aligned}$$

So we can get an upper bound for $D(T)$ from an upper bound for $\operatorname{Im} \int_{2+iT}^{2+i(T+1)} \frac{\xi'(s)}{\xi(s)}ds$. This is done in the next lemma.

We denote by $O(s^0)$ any function $f(s)$ that remains bounded as $|s| \to \infty$.

Lemma F For $T \ge 0$, we have

$$\int_{2+iT}^{2+i(T+1)} \frac{\xi'(s)}{\xi(s)}ds = \frac{i}{2}\log T + O(T^0).$$

Proof. We will use the Stirling's approximation

$$\Pi(s) = s^s e^{-s}\sqrt{2\pi s}\left(1 + \frac{O(s^0)}{s}\right).$$

From the definition of $\xi(s)$, we find

$$\xi(s) = \prod_{k=0}^{5} f_k(s)$$

where

$$\begin{aligned} f_0(s) &= \left(\frac{s}{2}\right)^{s/2} \\ f_1(s) &= (e\pi)^{-s/2} \\ f_2(s) &= \sqrt{\pi s} \\ f_3(s) &= (s-1) \\ f_4(s) &= \zeta(s) \\ f_5(s) &= \left(1 + \frac{O(s^0)}{s}\right) \end{aligned}$$

For any non-zero differentiable function $f(s)$, let $L(f)(s) = \dfrac{f'(s)}{f(s)}$. Then $L(fg) = L(f) + L(g)$. So we find

$$L(\xi)(s)) = \sum_{k=0}^{5} L(f_k)(s),$$

and we need to show that

$$\sum_{k=0}^{5} \int_{2+iT}^{2+i(T+1)} L(f_k)(s)ds = \frac{i}{2}\log T + O(T^0).$$

We will show that all terms in the sum with $k > 0$ are bounded as $T \to \infty$, while the term with $k = 0$ is $(i/2)\log T + O(T^0)$. For $k = 1$, we have

$$L\left((e\pi)^{-s/2}\right) = -\frac{1}{2}\log(e\pi)$$

and so

$$\int_{2+iT}^{2+i(T+1)} L(f_1)(s)ds = -\frac{i}{2}\log(e\pi)$$

For $k = 2$,

$$L(\sqrt{\pi s}) = \frac{1}{2s},$$

and

$$\int_{2+iT}^{2+i(T+1)} L(f_2)(s)ds = \frac{1}{2}\log\left(1 + \frac{i}{2+iT}\right) = O(T^0).$$

For $k = 3$,

$$L(s-1) = \frac{1}{s-1},$$

and

$$\int_{2+iT}^{2+i(T+1)} L(f_3)(s)ds = \log\left(1 + \frac{i}{1+iT}\right) = O(T^0).$$

For $k = 4$,

$$L(\zeta(s)) = \frac{d}{ds}\log\zeta(s)$$

and for $\mathrm{Re}(s) = 2$,

$$|\log\zeta(2+it)| = \left|\sum_p \sum_{n=1}^{\infty} \frac{1}{n}\frac{1}{p^{2n+nit}}\right| \leq \sum_p \sum_{n=1}^{\infty} \frac{1}{n}\frac{1}{p^{2n}} = \log\zeta(2),$$

so

$$\int_{2+iT}^{2+i(T+1)} L(f_4)(s)ds = O(T^0).$$

For $k = 5$,

$$\int_{2+iT}^{2+i(T+1)} L(f_5)(s)ds = \log\left(1 + \frac{O(s^0)}{s}\right)\Bigg|_{2+iT}^{2+i(T+1)} = O(T^0).$$

For $k = 0$,

$$\int L\left(\frac{s}{2}\right)^{s/2} = \frac{1}{2}s\log\left(\frac{s}{2}\right) = \frac{1}{2}s\log s - \frac{1}{2}s\log 2,$$

and

$$\frac{s\log 2}{2}\Bigg|_{2+iT}^{2+i(T+1)} = O(T^0).$$

So it remains to show that

$$\frac{1}{2}s\log s\Bigg|_{2+iT}^{2+i(T+1)} = \frac{1}{2}i\log T + O(T^0).$$

This is easily seen from the computation

$$\begin{aligned}
& (2+i(T+1))\log(2+i(T+1)) - (2+iT)\log(2+iT) \\
= \; & (2+iT)\log(2+i(T+1)) + i\log(2+i(T+1)) - (2+iT)\log(2+iT) \\
= \; & (2+iT)\log\left(1+\frac{i}{2+iT}\right) + i\log(2+i(T+1)) \\
= \; & O(T^0) + i\log(2+i(T+1)) \\
= \; & i\log T + O(T^0) \;\blacksquare
\end{aligned}$$

Combining Lemma F and (37), we find that for large T,

$$D(T) \leq \frac{\log T}{2\tan^{-1}(1/2)} + c \leq 1.1\log T + c$$

for some constant c, and so

$$D(T) \leq 2\log T \text{ for all large } T \ \blacksquare$$

5.11 Proof of Lemma 11

Note that $|F_h(\rho)| = |F_h(\bar{\rho})|$. Write $\rho = \beta + i\gamma$. We will consider the first two and the last two terms of (26) separately.

For the first two terms, we find

$$\begin{aligned} &\left|\sum_{\rho} \frac{x^\rho}{\rho+b}F_h(\rho) - \sum_{|\mathrm{Im}(\rho)|\leq h} \frac{x^\rho}{\rho+b}F_h(\rho)\right| = \left|\sum_{|\mathrm{Im}(\rho)|>h} \frac{x^\rho}{\rho+b}F_h(\rho)\right| \\ \leq\ & \sum_{\gamma>h} \frac{x^\beta}{\gamma}\left|F_h(\beta+i\gamma)\right| + \sum_{\gamma<-h} \frac{x^\beta}{|\gamma|}\left|F_h(\beta+i\gamma)\right| \\ =\ & \sum_{\gamma>h} \frac{x^\beta}{\gamma}\left|F_h(\beta+i\gamma)\right| + \sum_{\gamma>h} \frac{x^\beta}{\gamma}\left|F_h(\beta-i\gamma)\right| \\ =\ & 2\sum_{\gamma>h} \frac{x^\beta}{\gamma}\left|F_h(\beta-i\gamma)\right| \end{aligned}$$

Since $0 \leq \mathrm{Re}(\rho) \leq 1$, and $a > 1$, we have $a - \beta \geq a - 1 > 0$. Using Lemma 7, we have, for some constant K,

$$\begin{aligned} |F_h(\beta - i\gamma)| &= \left|\frac{1}{2\pi}\int_{a-ih}^{a+ih} \frac{x^{s-\beta+i\gamma}}{s-\beta+i\gamma}ds\right| \\ &= \left|\frac{1}{2\pi}\int_{a-\beta+i(\gamma-h)}^{a-\beta+i(\gamma+h)} \frac{x^t}{t}dt\right| \\ &\leq \frac{Kx^{a-\beta}}{(a-\beta+\gamma-h)\log x} \leq \frac{Kx^{a-\beta}}{(a-1+\gamma-h)\log x} \end{aligned}$$

and so

$$\sum_{\gamma>h} \frac{x^\beta}{\gamma}|F_h(\beta-i\gamma)| \leq \frac{Kx^a}{\log x}\sum_{\gamma>h} \frac{1}{\gamma(a-1+\gamma-h)}.$$

Let $T > 0$ be large enough so that (according to Lemma 25) $D(h) \leq 2\log h$ for $h \geq T$. By increasing T if necessary, we can also assume that $\log(T+j) \leq$

$(T+j)^{1/2}$ for all $j \geq 0$. Then if $h \geq T$, we have

$$\begin{aligned}
&\sum_{\gamma>h} \frac{1}{\gamma(a-1+\gamma-h)} = \sum_{j=0}^{\infty} \sum_{h+j<\gamma\leq h+j+1} \frac{1}{\gamma(a-1+\gamma-h)} \\
\leq\ & \sum_{j=0}^{\infty} \frac{2\log(h+j)}{(h+j)(a-1+j)} \leq 2\sum_{j=0}^{\infty} \frac{(h+j)^{1/2}}{(h+j)(j+a-1)} \\
\leq\ & 2\sum_{j=0}^{\infty} \frac{1}{(h+j)^{1/4}(h+j)^{1/4}(j+a-1)} \leq \frac{2}{h^{1/4}} \sum_{j=0}^{\infty} \frac{1}{(h+j)^{1/4}(j+a-1)}.
\end{aligned}$$

The last infinite series converges, and so we conclude that

$$\lim_{h\to\infty} \left| \sum_{\rho} \frac{x^\rho}{\rho+b} F_h(\rho) - \sum_{|\mathrm{Im}(\rho)|\leq h} \frac{x^\rho}{\rho+b} F_h(\rho) \right| = 0.$$

Before considering the last two terms of (26), we prove a lemma. We will use the fact that the harmonic numbers $H_n = \sum_{k=1}^{n} 1/k$ are of order $\log n$ as $n \to \infty$, so that we can write $H_n = \log n(1+\epsilon_n)$ where $\epsilon_n \to 0$ as $n \to \infty$.

Lemma G Denote by γ the imaginary part of a root of $\xi(s)$. Then

$$\lim_{h\to\infty} \frac{1}{h} \sum_{0<\gamma\leq h} \frac{1}{\gamma} = 0 \tag{38}$$

and if $c > 0$,

$$\lim_{h\to\infty} \frac{1}{h} \sum_{0<\gamma\leq h} \frac{1}{c+h-\gamma} = 0 \tag{39}$$

Proof. Let N be an integer large enough so that (according to Lemma 25) the number of roots in $[N+j, N+j+1]$ is at most $2\log(N+j)$, for all $j \geq 0$. We then find

$$\frac{1}{h} \sum_{0<\gamma\leq h} \frac{1}{\gamma} = \frac{1}{h} \sum_{0<\gamma\leq N} \frac{1}{\gamma} + \frac{1}{h} \sum_{N<\gamma\leq h} \frac{1}{\gamma}$$

and the limit of the first (finite) sum is 0 as $h \to \infty$. For the second sum, we find

$$\begin{aligned}
&\frac{1}{h} \sum_{N<\gamma\leq h} \frac{1}{\gamma} = \frac{1}{h} \sum_{j=0}^{\lfloor h-N \rfloor} \sum_{N+j<\gamma\leq N+j+1} \frac{1}{\gamma} \leq \frac{1}{h} \sum_{j=0}^{\lfloor h-N \rfloor} \frac{2\log(N+j)}{N+j} \\
\leq\ & \frac{2\log h}{h} \sum_{j=0}^{\lfloor h \rfloor} \frac{1}{N+j} \leq \frac{2\log h}{h} H_{N+\lfloor h \rfloor} = \frac{2\log h}{h} \log(N+h)(1+\epsilon_h).
\end{aligned}$$

Since $(\log h)^2/h \to 0$ as $h \to \infty$, this proves (38).

To prove (39), it is enough to consider (as for (38)) the sum for $N < \gamma \le h$. We find

$$\sum_{N<\gamma\le h} \frac{1}{c+h-\gamma} = \sum_{j=0}^{\lfloor h-N\rfloor -1} \sum_{N+j<\gamma\le N+j+1} \frac{1}{c+h-\gamma} + \sum_{N+\lfloor h-N\rfloor -1<\gamma\le h} \frac{1}{c+h-\gamma}.$$

The second sum is at most $(2\log h)/c$. The first sum is at most

$$\sum_{j=0}^{\lfloor h-N\rfloor -1} \frac{2\log h}{c+h-N-j-1} \le 2\log h\left(\frac{1}{c} + H_{\lfloor h-N\rfloor}\right) \le 2\log h\left(\frac{1}{c} + \log h(1+\epsilon_h)\right).$$

This proves (39) ■

We now consider the last two terms of (26). We have

$$\begin{aligned} F_h(\rho) - 1 &= \frac{1}{2\pi i}\int_{a-ih}^{a+ih} \frac{x^{s-\rho}}{s-\rho} ds - 1 = \frac{1}{2\pi i}\int_{a-\beta-i\gamma-ih}^{a-\beta-i\gamma+ih} \frac{x^t}{t} dt - 1 \\ &= \frac{1}{2\pi i}\int_{a-\beta-i(\gamma+h)}^{a-\beta+i(\gamma+h)} \frac{x^t}{t} dt - 1 - \frac{1}{2\pi i}\int_{a-\beta+i(h-\gamma)}^{a-\beta+i(h+\gamma)} \frac{x^t}{t} dt. \end{aligned}$$

Using the estimate (21) and Lemma 7, we find

$$\frac{x^\beta}{\gamma}|F_h(\rho) - 1| \le \frac{x^a}{\pi\log x}\frac{1}{\gamma(\gamma+h)} + \frac{Kx^a}{\log x}\frac{1}{\gamma(c+h-\gamma)}$$

$$= \frac{x^a}{\pi h\log x}\left(\frac{1}{\gamma} - \frac{1}{\gamma+h}\right) + \frac{Kx^a}{(c+h)\log x}\left(\frac{1}{\gamma} + \frac{1}{c+h-\gamma}\right)$$

$$\le \left(\frac{x^a}{\pi\log x} + \frac{Kx^a}{\log x}\right)\frac{1}{h\gamma} + \frac{Kx^a}{\log x}\frac{1}{h(c+h-\gamma)}.$$

So using (38) and (39) we find

$$\lim_{h\to\infty} \sum_{|\mathrm{Im}(\gamma)|\le h} \frac{x^\rho}{\rho+b}(F_h(\rho) - 1) = 0.$$

A Euler Product Formula

- Let $p_1, p_2, p_3, \ldots$ be the prime numbers, and $P_m = \{p_i : 1 \le i \le m\}$. Let s be a complex number with $\sigma = \mathrm{Re}(s) > 1$. Then

$$\sum_{j=1}^{\infty} \frac{1}{p_i^{sj}} = \left(1 - \frac{1}{p_i^s}\right)^{-1},$$

and so

$$\prod_{j=i}^{m}\left(1-\frac{1}{p_j^s}\right)^{-1} = \sum_{f\in\mathbb{Z}_+{}^m} \frac{1}{p_1^{sf(1)}p_2^{sf(2)}\cdots p_m^{sf(m)}}$$

and by the fundamental theorem of arithmetic, this last sum is $\displaystyle\sum_{n\in P_m}\frac{1}{n^s}$, that is absolutely convergent because dominated by the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^\sigma}$. So we can write

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{j=i}^{m}\left(1-\frac{1}{p_j^s}\right)^{-1} + \sum_{n\notin P_m}\frac{1}{n^s}.$$

If $n \in P_m$, then $n > m$, and so

$$\left|\sum_{n\notin P_m}\frac{1}{n^s}\right| \le \sum_{n=m}^{\infty}\frac{1}{n^\sigma} \to 0 \text{ as } m\to\infty.$$

So letting $m\to\infty$ in

$$\prod_{j=i}^{m}\left(1-\frac{1}{p_j^s}\right)^{-1} = \zeta(s) - \sum_{n\notin P_m}\frac{1}{n^s}$$

we conclude that

$$\prod_{j=i}^{\infty}\left(1-\frac{1}{p_j^s}\right)^{-1} = \zeta(s) \tag{A1}$$

and the convergence is uniform on $\mathrm{Re}(s) \ge a$ for any $a > 1$.

- Let B_n be as in the previous item. Then

$$\log\sum_{k\notin B_n}\frac{1}{k^s} = \sum_{i=1}^{n}\log\left(1-\frac{1}{p_i^s}\right)^{-1}, s > 1.$$

The sum on the right is

$$\sum_{i=1}^{n}\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{p_i^{ms}} = \sum_{i=1}^{n}\frac{1}{p_i^s} + \sum_{i=1}^{n}\sum_{m=2}^{\infty}\frac{1}{m}\frac{1}{p_i^{ms}}$$

and the second sum on the right is bounded by

$$\begin{aligned}\sum_{r=2}^{p_n}\sum_{m=2}^{\infty}\frac{1}{m}\frac{1}{r^{ms}} \le \frac{1}{2}\sum_{r=2}^{p_n}\sum_{m=2}^{\infty}\left(\frac{1}{r^s}\right)^m &= \frac{1}{2}\sum_{r=2}^{p_n}\frac{1}{r^{2s}}\frac{1}{1-r^{-s}} \\ &= \frac{1}{2}\sum_{r=2}^{\infty}\frac{1}{r^s(r^s-1)} \le \frac{1}{2}.\end{aligned}$$

So letting $n \to \infty$ we see that

$$\log \zeta(s) = \sum_{i=1}^{\infty} \frac{1}{p_i^s} + B(s) \text{ for } s > 1, \tag{A2}$$

where $|B(s)| \leq 1/2$ for $s \geq 1$. Hence the sum $\sum_{i=1}^{\infty} \frac{1}{p_i^s}$ diverges as $s \to 1^+$ like $\log \zeta(s)$.

B Extension of the factorial

Let $s \in \mathbb{C}$, and s not a negative integer. Define, for $n \geq 0$, $u_0 = 1$ and

$$u_n = \frac{n!(n+1)^s}{(s+1)(s+2)\cdots(s+n)}, n \geq 1.$$

Then

$$\frac{u_n}{u_{n-1}} = \frac{1}{1+\frac{s}{n}}\left(1+\frac{1}{n}\right)^s = \left(1-\frac{s}{n}+\frac{1}{n^2}a_n\right)\left(1+\frac{s}{n}+\frac{1}{n^2}b_n\right) = 1+\frac{c_n}{n^2},$$

where a_n, b_n, c_n are bounded as $n \to \infty$. So by standard results about infinite products [Appendix C] $\prod_{n=1}^{\infty} \frac{u_n}{u_{n-1}}$ converges absolutely. This means that

$$\lim_{n\to\infty} \prod_{k=1}^{n} \frac{u_k}{u_{k-1}} = \lim_{n\to\infty} u_n$$

exists for all $s \in \mathbb{C}$, s not a negative integer, and we define it to be $\Pi(s)$. So

$$\Pi(s) = \lim_{n\to\infty} \frac{n!(n+1)^s}{(s+1)(s+2)\cdots(s+n)}, s \neq -1, -2, -3, \ldots.$$

If $s = m$ is a positive integer, then

$$\begin{aligned} u_n = \frac{n!(n+1)^m}{(m+1)\cdots(m+n)} = m!\frac{n!(n+1)^m}{(m+n)!} &= m!\frac{(n+1)(n+1)\cdots(n+1)}{(n+1)(n+2)\cdots(n+m)} \\ &\to m! \text{ as } n \to \infty. \end{aligned}$$

So $\Pi(m) = m!$ and $\Pi(s)$ is an extension of the factorial. Some routine calculations show that

$$\Pi(s) = \prod_{k=1}^{\infty} \left(1+\frac{s}{k}\right)^{-1} \left(1+\frac{1}{k}\right)^s$$

C Infinite products

1. Suppose $s_k, k \geq 1$ are complex numbers, and let $p_n = s_1 s_2 \cdots s_n = \prod_{k=1}^{n} s_k$.

   (a) If $\lim_{n\to\infty} p_n$ exists and is non-zero, we denote it by $\prod_{k=1}^{\infty} s_k$, we call it the *infinite product* of the sequence s_k, and say that the infinite product *converges.*

   In this case, clearly we must have $s_k \neq 0$ for all k, and if we let $p = \prod_{k=1}^{\infty} s_k$, we have

   $$s_n = \frac{p_n}{p_{n-1}} \to \frac{p}{p} = 1 \text{ as } n \to \infty.$$

   So a necessary condition for the infinite product $\prod_{k=1}^{\infty} s_k$ to converge is that $s_k \to 1$ as $k \to \infty$.

   (b) If $s_k \neq 0$ for all k but $\lim_{n\to\infty} p_n = 0$, we say that the infinite product *diverges to zero*. An example is the case $s_k = a$ for all k, where $|a| < 1$.

   (c) If $s_k = 0$ for some k, but there is some m such that $s_k \neq 0$ for $k \geq m$, and the infinite product $s_m s_{m+1} s_{m+2} \cdots$ converges, we say that the infinite product *converges to zero*. Note that in this terminology (used for example by Whittaker & Watson in *A course of modern analysis*) an infinite product that converges to zero does not converge. Other sources will say that the infinite product diverges as soon as $\lim_{n\to\infty} p_n = 0$.

2. Since

   $$\log \prod_{k=1}^{n} (1 + s_k) = \sum_{k=1}^{n} \log(1 + s_k),$$

   we see from the definitions that

   $$\prod_{n=1}^{\infty} (1 + s_n) \text{ converges } \iff \sum_{n=1}^{\infty} \log(1 + s_n) \text{ converges }.$$

   Note that if $\sum_{n=1}^{\infty} \log(1 + s_n)$ converges, then

   $$\prod_{n=1}^{\infty} (1 + s_n) = \exp\left(\sum_{n=1}^{\infty} \log(1 + s_n)\right) \neq 0.$$

3. Let s_n be a sequence of complex numbers, with $s_n \neq -1$ for all n. So $\log(1+s_n)$ is defined for all n. Suppose $s_n \to 0$ as $n \to \infty$. Find m such that $|s_k| \leq 1/2$ for $k \geq m$. Then we find, for $k \geq m$,

$$\log(1+s_k) = s_k \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} s_k^n\right).$$

Since

$$\left|\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} s_k^n\right| \leq \sum_{n=1}^{\infty} \frac{1}{n+1} |s_k|^n \leq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2},$$

we find

$$\frac{1}{2}|s_k| \leq |\log(1+s_k)| \leq \frac{3}{2}|s_k|.$$

So by the comparison test we see that $\sum_{n=1}^{\infty} s_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} \log(1+s_n)$ converges absolutely.

4. We say that the infinite product $\prod_{n=1}^{\infty}(1+s_n)$ *converges absolutely* if the infinite series $\sum_{n=1}^{\infty} \log(1+s_n)$ converges absolutely. If $I_+ = \{i : s_i \geq 0\}$ and $I_- = \{i : s_i < 0\}$, absolute convergence of $\prod_{n=1}^{\infty}(1+s_n)$ means that both $\prod_{i \in I_+}(1+s_i)$ and $\prod_{i \in I_-}(1+s_i)$ converge. By the previous item, $\prod_{n=1}^{\infty}(1+s_n)$ converges absolutely if and only if $\sum_{n=1}^{\infty} s_n$ converges absolutely.

D The functional equation for the theta function

Jacobi's *theta function* $\vartheta(z;\tau)$ is defined for $z \in \mathbb{C}$ and $\text{Im}\tau > 0$ to be

$$\vartheta(z;\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z}.$$

Consider the special case

$$G(u) = \vartheta(0; iu^2) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 u^2}, u > 0$$

We prove that G satisfies the functional equation

$$G(u) = \frac{1}{u} G\left(\frac{1}{u}\right). \tag{D1}$$

Let $u > 0$ be fixed, and consider the function

$$f(x) = e^{-\pi x^2/u^2}.$$

Let

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{2\pi i x s} dx$$

be its Fourier transform. Then by Parseval's theorem [E3]

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

We find

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2/u^2} = G\left(\frac{1}{u}\right).$$

Using the integral formula

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$$

and Cauchy's theorem to change the path of integration from $(-\infty + iu, \infty + iu)$ to $(-\infty, \infty)$, we also find

$$\begin{aligned} \hat{f}(n) &= \int_{-\infty}^{\infty} e^{-\pi x^2/u^2} e^{2\pi i x n} dx = u \int_{-\infty}^{\infty} e^{-\pi t^2} e^{2\pi i u n t} dt = u e^{-\pi u^2 n^2} \int_{-\infty}^{\infty} e^{-\pi (t - iu)^2} dt \\ &= u e^{-\pi u^2 n^2} \int_{-\infty + iu}^{\infty + iu} e^{-\pi z^2} dz = u e^{-\pi u^2 n^2} \int_{-\infty}^{\infty} e^{-\pi x^2} dx = u e^{-\pi u^2 n^2}. \end{aligned}$$

So we conclude that

$$\sum_{n \in \mathbb{Z}} \hat{f}(n) = u G(u),$$

and so by Parseval's theorem (E3) the functional equation (D1) follows.

E Parseval's theorem

Let $f : \mathbb{R} \to \mathbb{C}$ be a function that goes to zero sufficiently fast at $\pm\infty$. The *Fourier transform* of f is

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x) e^{2\pi i x u} dx.$$

We can "periodify" f by defining

$$F(x) = \sum_{n \in \mathbb{Z}} f(x+n). \tag{E1}$$

Then by the theory of Fourier series [see Rudin, Real and Complex Analysis],

$$F(x) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x}, \tag{E2}$$

where

$$a_n = \int_0^1 F(x) e^{-2\pi i n x} dx.$$

We then find

$$\begin{aligned} a_n &= \int_0^1 \sum_{m \in \mathbb{Z}} f(x+m) e^{-2\pi i n x} dx \\ &= \sum_{m \in \mathbb{Z}} \int_0^1 f(x+m) e^{-2\pi i n x} dx \\ &= \sum_{m \in \mathbb{Z}} \int_m^{m+1} f(t) e^{-2\pi i n (t-m)} dt \\ &= \sum_{m \in \mathbb{Z}} \int_m^{m+1} f(t) e^{-2\pi i n t} dt \\ &= \int_{-\infty}^{\infty} f(t) e^{-2\pi i n t} dt = \hat{f}(-n). \end{aligned}$$

Setting $x = 0$ in (E1) and (E2), we find

$$\sum_{n \in \mathbb{Z}} f(n) = F(0) = \sum_{n \in \mathbb{Z}} a_n = \sum_{n \in \mathbb{Z}} \hat{f}(-n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

This proves Parseval's Theorem:

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n). \tag{E3}$$

F Jensen's formula

Suppose that f is analytic on an open set containing $\overline{B}(0, R)$, and $f(z) \neq 0$ in $B(0, R)$. Integrating the Maclaurin series for f we find

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(Re^{i\theta}\right) d\theta.$$

Note that for any $w \neq 0$ we have $\text{Re}(\log w) = \log |w|$. So applying the above formula to $\log(f(z))$ and taking the real part, we find

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(Re^{i\theta}) \right| d\theta.$$

Now suppose f has a simple zero at $z_0 \in B(0, R)$, and $z_0 \neq 0$. If we can find some $\phi(z)$ with a simple pole at z_0, without zeros for $|z| \leq R$, and such that $|\phi(z)| = 1$ on $|z| = R$, then the function $f(z)\phi(z)$ has no zeros in $B(0, R)$, and since $|f(z)\phi(z)| = |f(z)|$ for $|z| = R$, applying the above formula to $f(z)\phi(z)$ we have

$$\log(|f(0)\phi(0)|) = \frac{1}{2\pi} \int_0^{2\pi} \log(|f(Re^{i\theta})| d\theta.$$

Such a factor is

$$\phi(z) = \frac{R^2 - \overline{z_0} z}{R(z - z_0)},$$

as we can see by noting that $R^2 - \overline{z_0} z \neq 0$ in $B(0, R)$, and for $|z| = R$,

$$\left| \frac{R^2 - \overline{z_0} z}{R(z - z_0)} \right| = \left| \frac{(R^2 - \overline{z_0} z)\overline{z}}{R(z - z_0)\overline{z}} \right| = \left| \frac{R^2(\overline{z} - \overline{z_0})}{R^2(z - z_0)} \right| = 1.$$

Since $\phi(0) = R/|z_0|$, we find

$$\log \left(|f(0)| \frac{R}{|z_0|} \right) = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(Re^{i\theta}) \right| d\theta.$$

In the same way, if $f(0) \neq 0$ and f has zeros $z_1, \ldots, z_k$ in $B(0, R)$ (repeated according to multiplicity), consider $f(z) \dfrac{R^2 - \overline{z_1} z}{R(z - z_1)} \cdots \dfrac{R^2 - \overline{z_k} z}{R(z - z_k)}$. We then find Jensen's formula:

$$\log \left(|f(0)| \frac{R}{|z_1|} \cdots \frac{R}{|z_k|} \right) = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(Re^{i\theta}) \right| d\theta.$$

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